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University of Heidelberg

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Philipp Heinen

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Phase Correlators

in Gross-Pitaevskii Systems

Far from Equilibrium

This Master thesis has been carried out by Philipp Heinen

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under the supervision of

Herrn Prof. Thomas Gasenzer

Phasenkorrelatoren in Gross-Pitaevskii-Systemen fernab des Gleichgewichts:

Wir diskutieren verschiedene Ansätze zur Definition einer Korrelationsfunktion von Phasenfeldern und untersuchen deren Eigenschaften in einem eindimensionalen Spin-System. Wir kommen zu dem Schluss, dass die natürlichste Art, einen solchen Korrelator zu definieren, darin besteht, einen Schallwellen- und Vortex-Anteil der Phase als skalares beziehungsweise Vektor-Potential des Geschwindigkeitsfeldes einzuführen. Basierend auf dieser Definition extrahieren wir numerisch Phasenkorrelatoren in einem zweidimensionalen $U(N)$-symmetrischen Gross-Pitaevskii-System. Der Schallwellen-Anteil wird mit den Vorhersagen aus einer effektiven Feldtheorie verglichen. Bezüglich des Vortex-Anteils argumentieren wir, warum eine näherungsweise Dualität zwischen Vortex-Dynamik und dem 2+1D Sine-Gordon-Modell gelten könnte. Im letzten Teil entwickeln wir eine kinetische Theorie dieses Modells, aus welcher wir einen dynamischen Skalierungsexponenten $\beta = 1/6$ finden. Dieses Ergebnis wird untermauert durch numerische Simulationen. Damit geben wir eine mögliche Erklärung für das anomale $\beta \sim 0.2$ in vortexdominierten Systemen.

Phase correlators in Gross-Pitaevskii systems far from equilibrium:

We discuss different approaches of defining a correlation function of phase fields and study their properties in a one-dimensional spin system. We conclude that the most natural way of defining such a correlator is to introduce a sound-wave and vortex part of the phase as the scalar and vector potential of the velocity field. Based on this definition we numerically extract phase correlators in a two-dimensional $U(N)$ symmetric Gross-Pitaevskii system. The sound wave part is compared to predictions from a low-energy effective field theory. Regarding the vortex part we argue why an approximate duality between vortex dynamics and the 2+1D Sine-Gordon model might hold. In the last part we develop a kinetic theory of this model, from which we find a dynamical scaling exponent $\beta = 1/6$. This result is further corroborated by numerical simulations. Thereby a possible explanation for the anomalous $\beta \sim 0.2$ in vortex dominated systems is provided.

Contents

1 Introduction

Quantum many-body systems far from thermal equilibrium are an intriguing as well as tempting field of study. A plethora of new phenomena show up on the way back to thermal equilibrium, such as parametric resonance, turbulence, prethermalization, prescaling and non-thermal fixed points [1–6]. These occur in a similar way in very different scenarios, rendering non-equilibrium quantum theory applicable to a broad range of physical systems, such as early universe dynamics, heavy ion collisions and ultra-cold quantum gases [7].

This thesis mainly deals with the non-equilibrium dynamics of the non-relativistic Gross-Pitaevskii model, though in the last chapters we will also have a look at the relativistic Sine-Gordon model. The particular focus will be on the phase θ of the complex Gross-Pitaevskii field ψ, motivated by the fact that at low energies fluctuations in the density $\rho = |\psi|^2$ are suppressed and the phases become the relevant degree of freedom [8]. We will therefore study correlation functions of the phases numerically, comparing to existing analytical predictions, as well as analytically, aiming at giving a possible explanation for numerical results not fully understood yet.

The work is organized as follows: in the remainder of this chapter, we will introduce the Gross-Pitaevskii model and briefly discuss some of its properties. Furthermore we discuss the notion of universality far from equilibrium and non-thermal fixed points. The next chapter gives a short overview over the techniques of non-equilibrium quantum field theory. After these two introductory chapters we discuss in detail several possibilities to introduce a correlation function of phase variables (chapter 3). Thereafter these different definitions are applied to a one-dimensional spin system and discussed in this context (chapter 4). We will then conclude that the most reasonable way of defining phase correlators is to correlate either the scalar or the vector potential (the "sound wave" and the "vortex" part, respectively) of the velocity field $\mathbf{V}$ of the Gross-Pitaevskii field ψ. This definition will be the only one considered in the remainder of the thesis. Based on this definition, chapter 5 discusses several aspects of the phase representation of Gross-Pitaevskii theory. We review earlier findings concerning the "sound wave" part of the phase. Furthermore we aim at giving also a description for the "vortex" part. To this end we propose arguments why, under certain assumptions, the well-known duality between vortices and the Sine-Gordon model in 2D might approximately persist to the 2+1D case. Finally this chapter deals with the relation between correlation functions of the fundamental field ψ and of the phase fields, in part reviewing again earlier findings but including now also the "vortex" part. The following chapter (6) deals with the numerical computation of phase correlators in a two-dimensional $U(1)$ and $U(3)$ symmetric system. Thereafter we discuss the non-equilibrium dynamics of the Sine-Gordon model. Whereas in chapter 7 we make the attempt of developing a kinetic description based on the 2PI effective action formalism and the Boltzmann equation, chapter 8 gives a numerical treatment employing the truncated Wigner approximation. Finally in chapter 9 we summarize the results and give an outlook.

1.1 Gross-Pitaevskii model and vortices

The central model studied in this thesis is the $U(N)$ symmetric Gross-Pitaevskii (GP) model whose Lagrangian is given by

$$\mathcal{L}_{\mathrm{GP}} = \frac{i}{2}(\psi_a^* \partial_t \psi_a - \psi_a \partial_t \psi_a^*) - \frac{1}{2m}\nabla\psi_a^* \cdot \nabla\psi_a - \frac{g}{2}(\psi_a^*\psi_a)^2 \tag{1.1}$$

where ψ_a is an N-component scalar Bose field and summation over indices is implied. For the important case of $N = 1$ this reduces to

$$\mathcal{L}_{\mathrm{GP}} = \frac{i}{2}(\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi - \frac{g}{2}|\psi|^4. \tag{1.2}$$

This Lagrangian provides an effective description of a Bose-Einstein condensate of ultra-cold atoms (cf. e. g. [9]), with the identification

$$g = \frac{4\pi a}{m} \tag{1.3}$$

with a the scattering length of the atoms and m their mass. For $N > 1$ one can see the different field components e. g. as representations of atoms in different hyperfine or Zeeman states, or of atoms of different elements [10]. Apart from the description of ultra-cold atoms, there exists also an interpretation of the Gross-Pitaevskii Lagrangian as the effective description of relativistic scalar field theories at low momenta [11].
The classical equations of motion can be easily obtained as the Euler-Lagrange equations of the Lagrangian (1.1):

$$i\partial_t \psi_a = \left(-\frac{1}{2m}\Delta + g(\psi_b^*\psi_b)\right)\psi_a \tag{1.4}$$

and in the one-component case

$$i\partial_t \psi = \left(-\frac{1}{2m}\Delta + g|\psi|^2\right)\psi. \tag{1.5}$$

As for the wave function in the Schrödinger equation one can define a density and a current for the Gross-Pitaevskii field, given by

$$\rho = |\psi|^2 \tag{1.6}$$

$$\mathbf{j} = \frac{1}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{1.7}$$

It is easy to show that also in Gross-Pitaevskii theory the continuity equation $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$ is fulfilled.
An important solution to the two-dimensional Gross-Pitaevskii equation is the *vortex* solution, which is topologically non-trivial. It is characterized by a drop of the density ρ to 0 in the vortex core and a winding of the phase around it. We make the ansatz

$$\psi(r, \varphi; t) = \sqrt{\rho_0} f(r) e^{iq\varphi} e^{-i\omega t} \tag{1.8}$$

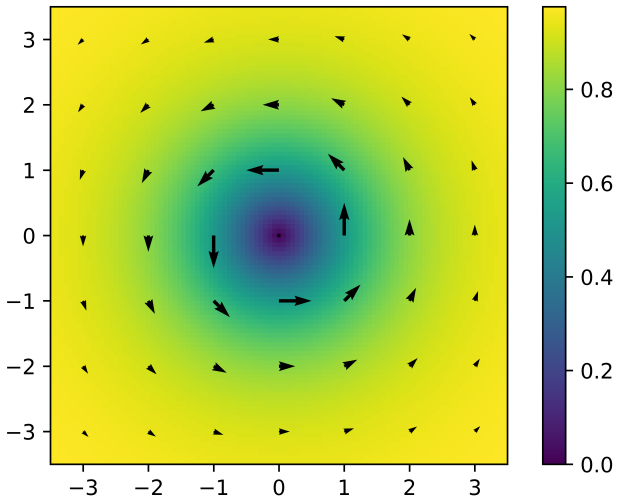

Figure 1.1: Density profile (normalized to the bulk density) and velocity field of a vortex.

with $q \in \mathbb{Z}$ the integer "charge" of the vortex and ρ_0 the bulk density outside the vortex. Inserting this into the Gross-Pitaevskii equation yields

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{q^2}{r^2} f + 2m\omega f - 2mg\rho_0 f^3 = 0. \tag{1.9}$$

For $r \to \infty$ we should have $f \to 1$. This enforces $\omega = g\rho_0$, so that the equation becomes

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{q^2}{r^2} f + 2mg\rho_0(f - f^3) = 0. \tag{1.10}$$

It turns out that there is a characteristic length scale $\xi_h = 1/\sqrt{2mg\rho_0}$. Indeed by introducing the dimensionless variable $x = r/\xi_h$ we can write the equation for f in a dimensionless form:

$$f'' + \frac{1}{x} f' - \frac{q^2}{x^2} f + f - f^3 = 0. \tag{1.11}$$

ξ_h is called *healing length* because it is the length scale on which the defect in the density is "healed", i. e. on which the density returns from $\rho = 0$ in the vortex center to the bulk value $\rho = \rho_0$. Equation (1.11) has no simple analytic solution but a numerical evaluation is straightforward.
For the velocity field $\mathbf{v} \equiv \mathbf{j}/\rho$ of a vortex one finds

$$\mathbf{v} = \frac{q}{mr}\mathbf{e}_\varphi. \tag{1.12}$$

Figure (1.1) shows the density profile (as calculated numerically from (1.11)) and the velocity field of a vortex together.
Vortices are very robust objects and can act as quasiparticles that move in space, scatter from each other, form bound pairs and annihilate. In many situations they are essential to the dynamics of a Gross-Pitaevskii system in 2D [12]. An important part of this thesis will deal with a particular attempt to describe their dynamical behavior.

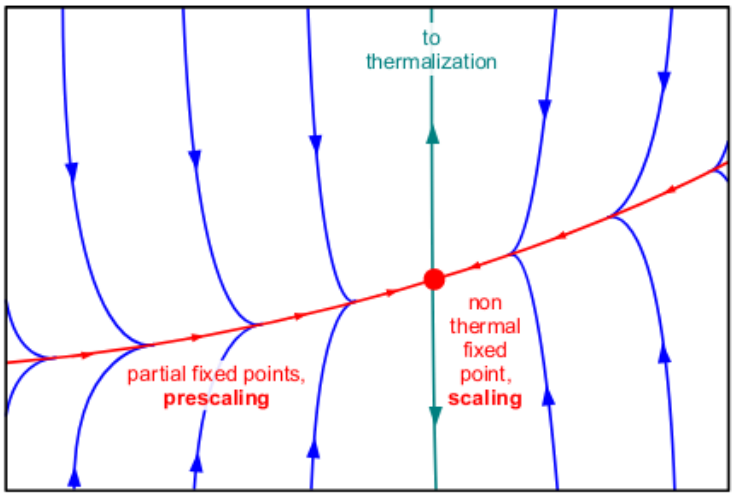

Figure 1.2: Sketch of the flow with time in parameter space. Picture taken from [5].

In 3D, there cannot be point-like topologically non-trivial objects since the GP field ψ has only two components (if one considers its real and imaginary part as independent components). However, tube-like topological defects are still possible, which are frequently called *vortex tubes*. Indeed they occur in three-dimensional GP systems and can play an important role for the dynamics too, as their two-dimensional counterparts [12]. However, as we are mainly concerned with two-dimensional systems, we will not discuss vortex tubes in more detail here.

1.2 Non-thermal fixed points and scaling

Whereas the notion of *universality* is well-established for systems in thermal equilibrium, universality in systems far from thermal equilibrium is a rather recent subject of study. By universality one understands the fact that very different systems exhibit very similar behavior near a phase transition. This manifests itself in a set of *critical exponents* that depend only on very general properties such as the dimension and symmetries of a system. A celebrated explanation for this phenomenon is provided by renormalization group theory (see e. g. [13]). Here one studies the flow in an effective parameter space of a theory under the change of a flow parameter λ, which encodes the spatial resolution on which the system is looked at. It then turns out that there are *fixed points* in the parameter space to which the system flows from a broad range of initial parameters if the spatial resolution is sent to 0. Thus they account for the very similar behavior of very different systems near phase transitions, where spatial coherence typically grows large and hence the spatial resolution becomes small.
The notion of *non-thermal* fixed points [14] arises from applying this idea from the theory of thermal equilibrium systems to the non-equilibrium case. Instead of the spatial resolution the role of the flow parameter λ is now played by the time t itself. Such a flow with time in the effective parameter space of a theory is sketched in figure (1.2).
One of the most frequently considered observables in the study of non-thermal fixed points is the occupation number in momentum space $f(t;k)$. One postulates that at a

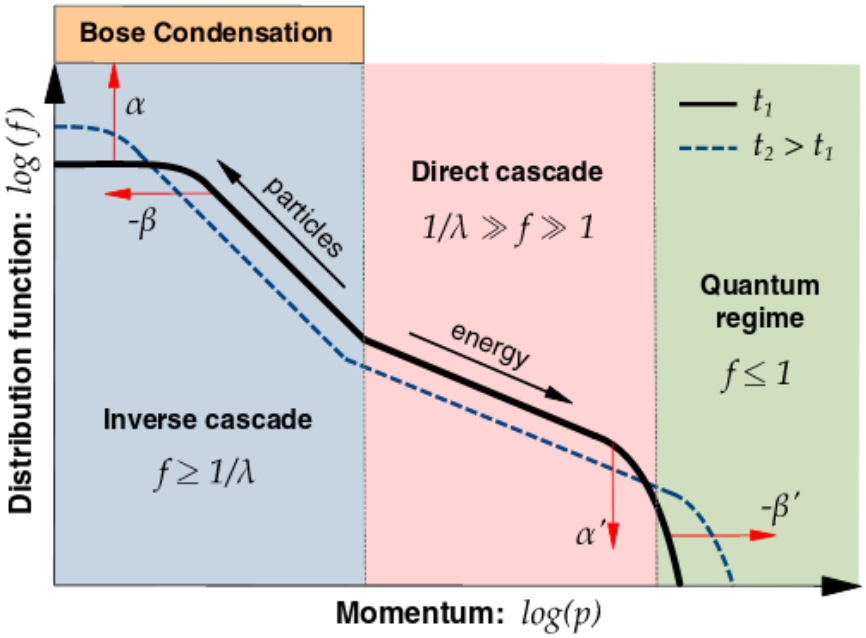

Figure 1.3: Sketch of $f(k)$ at a non-thermal fixed point. Picture taken from [15].

non-thermal fixed point the occupation number fulfills the following relation:

$$f(t;k) = (t/t_0)^{\alpha} f_S((t/t_0)^{\beta} k) \tag{1.13}$$

where t_0 is a reference time, $f_S(k) \equiv f(t = t_0; k)$ is the scaling function and α and β are scaling exponents. Predicting their values is a central issue in the theory of non-thermal fixed points, such as predicting critical exponents is one of the most important problems in equilibrium theory. α and β are not entirely independent if one assumes a conservation law to hold. E. g. if one assumes particle number conservation, the quantity

$$\int d^d k f(t;k) \tag{1.14}$$

has to remain constant from which it follows that $\alpha = d\beta$. Similarly, energy conservation implies the conservation of the integral

$$\int d^d k f(t;k)\omega_k \tag{1.15}$$

with ω_k the dispersion relation. If $\omega_k \sim k^z$ this implies $\alpha = (d+z)\beta$. Typically, one observes the fulfillment of particle conservation in the IR part of the spectrum whereas energy conservation holds in the UV. The exponents α and β are thus different in the IR and UV. Usually they have even different signs. Whereas in the IR β is typically positive, implying a transport of particles towards lower momenta, in the UV we have a movement into the opposite direction, i. e. towards higher momenta [5]. This is depicted schematically in figure (1.3). In this thesis we will be concerned only with the dynamics in the IR, thus in the following, whenever we talk about exponents α and β, we are referring to those of the IR part of the spectrum. In this regime, relativistic systems with a mass gap and non-relativistic systems typically show very similar behavior and are characterized by the same exponents because at low momenta the relativistic theory becomes effectively

non-relativistic [15].
Regarding the shape of the function $f(k)$ one typically finds that at low momenta $f(k)$ shows a plateau and then falls off with a characteristic power law $k^{-\kappa}$ [5]. In the UV there can be additional structure. The point where the plateau forms indicates a characteristic momentum scale k_{char} and thus a characteristic length scale ξ_{char} in the system. The value of the spatial scaling exponent κ is also an object of study that one tries to predict analytically.
One of the most important predictions coming from the non-equilibrium QFT formalism outlined in the next chapter is that $\beta = 1/2$. This value is indeed found in a plethora of different systems, relativistic and non-relativistic ones alike. However, there are exceptions, one of the most important ones being the one-component two-dimensional Gross-Pitaevskii system. Here one finds $\beta \sim 0.2$ [16]. Similar results also hold for one-component two-dimensional ϕ^4 theory, which in its IR modes can be approximately described by the GP system [11]. One can show that this deviation is caused by vortices, i. e. topological defects in the field configurations of such systems. While in [16] a heuristic argument for $\beta \sim 0.2$ is provided, there is no satisfying field-theoretic explanation so far. In a later chapter we will propose one possibility for such an explanation.

2 Non-equilibrium quantum field theory

In this section we want to summarize briefly the main concepts of non-equilibrium quantum field theory. For a more comprehensive treatment the reader is referred to [7], which we will follow closely.

2.1 Schwinger-Keldysh formalism

The majority of quantum field theoretic calculations does not concern the most general physical setup, i. e. a system in an arbitrary state allowed by the theory, but restricts itself to important special cases. In particle physics one usually performs computations in "zero-temperature" quantum field theory. This means one assumes the system to be in its ground state (the vacuum state) and considers excitations on top of this ground state. "Finite-temperature" QFT is more general, taking the system's density matrix to be a thermal one with an arbitrary temperature. This approach is chosen e. g. in condensed matter physics or the study of quark-gluon plasma. However, for treating systems out of thermal equilibrium one needs a still more general formalism. This is known under the name of Schwinger-Keldysh formalism.
Assume we have a system which at the initial time t_0 is described by the density matrix ρ_0. The expectation value of an operator $\mathcal{O}$ at time t is then given by

$$\mathrm{Tr}\left\{\rho_0 e^{iH(t-t_0)}\mathcal{O}e^{-iH(t-t_0)}\right\}. \tag{2.1}$$

This can be interpreted as propagating from t_0 to t and then from t back to t_0. Additionally one usually inserts a $\mathbf{1}$ for convenience, i. e. a propagation from t to ∞ and from ∞ back to t. This gives rise to the *Schwinger-Keldysh contour* $\mathcal{C}$. $\mathcal{C}$ runs from t_0 to ∞ and then back to t_0, see figure (2.1).
One can define then a non-equilibrium generating functional, from which arbitrary correlation functions can be calculated:

$$Z[J,R] = \mathrm{Tr}\left\{\rho_0 T_{\mathcal{C}} \exp\left(i\int_{x,\mathcal{C}} J(x)\phi(x) + i\frac{1}{2}\int_{xy,\mathcal{C}} \phi(x)R(x,y)\phi(y)\right)\right\} \tag{2.2}$$

where $\phi(x)$ is a Heisenberg field operator and $T_{\mathcal{C}}$ denotes time ordering along the Schwinger-Keldysh contour. E. g. one obtains from Z:

$$\left.\frac{\delta Z}{i\delta J(x)}\right|_{J,R=0} = \langle\phi(x)\rangle \tag{2.3}$$

$$\left.\frac{\delta^2 Z}{i\delta J(x) i\delta J(y)}\right|_{J,R=0} = \langle T_{\mathcal{C}}\phi(x)\phi(y)\rangle. \tag{2.4}$$

One further defines the connected propagator G as

$$G(x,y) = \langle T_{\mathcal{C}}\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle. \tag{2.5}$$

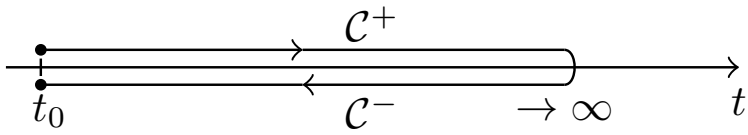

Figure 2.1: Schwinger-Keldysh contour $\mathcal{C}$ with upper ($\mathcal{C}^+$) and lower branch ($\mathcal{C}^-$).

(2.2) can be transformed into a non-equilibrium path integral:

$$Z[J,R] = \int [d\phi_0^+][d\phi_0^-]\langle\phi_0^+|\rho_0|\phi_0^-\rangle \times \int_{\phi_0^+}^{\phi_0^-} D\phi \exp\left[i\left(S[\phi] + \int_{x,\mathcal{C}} J(x)\phi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} \phi(x)R(x,y)\phi(y)\right)\right] \tag{2.6}$$

with $|\phi_0^\pm\rangle$ being eigenstates of the field operator at t_0, i. e.

$$\hat{\phi}(t_0,\mathbf{x})|\phi_0^\pm\rangle = \phi_0^\pm(\mathbf{x})|\phi_0^\pm\rangle \tag{2.7}$$

and

$$\int [d\phi_0^+][d\phi_0^-] \equiv \int \prod_{\mathbf{x}} d\phi_0^+(\mathbf{x}) d\phi_0^-(\mathbf{x}). \tag{2.8}$$

The notation $\int_{\phi_0^+}^{\phi_0^-} D\phi$ means that one integrates only over field configurations fulfilling the boundary condition $\phi(t_0^+,\mathbf{x}) = \phi_0^+(\mathbf{x})$ and $\phi(t_0^-,\mathbf{x}) = \phi_0^-(\mathbf{x})$ where $t_0^\pm$ denotes t_0 on the lower or upper branch of the Schwinger-Keldysh contour $\mathcal{C}$, respectively.
$\langle\phi_0^+|\rho_0|\phi_0^-\rangle$ can be parameterized as

$$\langle\phi_0^+|\rho_0|\phi_0^-\rangle = \mathcal{N}\exp\left\{i(h_1[\phi_0^+,\phi_0^-] + h_2[\phi_0^+,\phi_0^-] + \dots)\right\} \tag{2.9}$$

with

$$h_1[\phi_0^+,\phi_0^-] = \int d^dx\{\alpha_1^+(\mathbf{x})\phi_0^+(\mathbf{x}) + \alpha_1^-(\mathbf{x})\phi_0^-(\mathbf{x})\} \tag{2.10}$$

$$h_2[\phi_0^+,\phi_0^-] = \int d^dx d^dy\{\alpha_2^{++}(\mathbf{x},\mathbf{y})\phi_0^+(\mathbf{x})\phi_0^+(\mathbf{y}) + \alpha_2^{+-}(\mathbf{x},\mathbf{y})\phi_0^+(\mathbf{x})\phi_0^-(\mathbf{y}) + \alpha_2^{-+}(\mathbf{x},\mathbf{y})\phi_0^-(\mathbf{x})\phi_0^+(\mathbf{y}) + \alpha_2^{--}(\mathbf{x},\mathbf{y})\phi_0^-(\mathbf{x})\phi_0^-(\mathbf{y})\} \tag{2.11}$$

$$\dots$$

A very common assumption is that one can indeed truncate the series in (2.9) at second order, i. e. that the initial density matrix is Gaussian. This is fulfilled to a good accuracy in many systems. Then one can absorb $\langle\phi_0^+|\rho_0|\phi_0^-\rangle$ entirely into the source terms $J(x)$ and $R(x,y)$ so that these encode the initial conditions.

2.2 2PI effective action and $1/N$ expansion

Standard perturbation theory techniques are not suited very well for non-equilibrium problems. The reason are so-called secular terms growing exponentially in time, which

render a perturbative expansion invalid even for arbitrarily weak coupling at sufficiently late times [17].
One of the major approaches to circumvent this problem are n-particle irreducible (nPI) effective actions, in particular 2PI effective actions, generalizing the well-known concept of 1PI effective actions. Whereas the 1PI effective action is expressed in terms of the full quantum field ϕ, the 2PI effective action is expressed in terms of the full quantum field ϕ and the full propagator G. Correspondingly, the nPI effective action is expressed in terms of all k-point functions with $k \leq n$. However, for most non-equilibrium problems the 2PI formalism is sufficient.
As an example we consider relativistic ϕ^4 theory with the action

$$S[\phi] = \int_{x,\mathcal{C}} \left\{ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}. \tag{2.12}$$

The 2PI effective action is constructed by a twofold Legendre transform. First we define the function W by

$$\exp(iW[J]) \equiv Z[J,R]. \tag{2.13}$$

With this we can define

$$\Gamma[\phi, G] = W[J,R] - \int_{x,\mathcal{C}} \frac{\delta W}{\delta J(x)} J(x) - \int_{xy,\mathcal{C}} \frac{\delta W}{\delta R(x,y)} R(x,y). \tag{2.14}$$

Since $\Gamma[\phi, G]$ is a functional only of ϕ and G, J and R have to be expressed in terms of these new variables by resolving the equations

$$\frac{\delta \Gamma}{\delta \phi(x)} = -J(x) - \int_{y,\mathcal{C}} R(x,y)\phi(y) \tag{2.15}$$

$$\frac{\delta \Gamma}{\delta G(x,y)} = -\frac{1}{2} R(x,y). \tag{2.16}$$

At the same time, (2.15) and (2.16) are the full quantum equations of motion for ϕ and G. Given the currents J and R, they determine their dynamics entirely.
Γ can be calculated by standard loop expansion techniques. One then usually writes it as the sum of the 1-loop contribution and a "rest", i. e.

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\{\ln G^{-1}\} + \frac{i}{2}\mathrm{Tr}\{G_0^{-1}(\phi)G\} + \Gamma_2[\phi, G] + \text{const} \tag{2.17}$$

with the first three terms the one loop contribution and $\Gamma_2[\phi, G]$ the rest. G_0 is the classical propagator, i. e. $iG_0^{-1}(x,y) = \delta^2 S/\delta\phi(x)\delta\phi(y)$. The trace and the logarithm have to be understood in the functional sense, i. e. viewing $G(x,y)$ as a matrix with indices x and y and the logarithm being the matrix logarithm. From $\Gamma_2[\phi, G]$ one can define the self-energy as

$$\Sigma(x,y) \equiv 2i \frac{\delta \Gamma_2}{\delta G(x,y)}. \tag{2.18}$$

The perturbative computation of Γ_2 can be organized in terms of Feynman diagrams. Assuming the absence of symmetry breaking, i. e. $\phi = 0$, a ϕ^k term in the action with

coupling constant $\lambda_k/k!$ yields only k-vertices coming with a factor of $-i\lambda_k/k!$. These have to be connected by propagators $G(x,y)$ such that cutting through two arbitrary propagators does not make the diagram fall apart, i. e. one keeps only two-particle irreducible (2PI) contributions. Additionally one has to include an overall factor of $1/n!$ with n the number of vertices in a diagram (this factor stems from the expansion of the exponential). If $\phi \neq 0$, there are additional vertices containing also the field value ϕ. However, in this thesis we will not deal with this case so that we do not consider it further.
In non-equilibrium quantum field theory it is a common practice to decompose the propagator into a spectral component (ρ) and a statistical component (F). The former encodes the spectrum of the theory, i. e. the available states, whereas the latter describes their occupancy. This decomposition works as follows:

$$G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0). \tag{2.19}$$

Here $\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0)$ denotes the signum function on the Schwinger-Keldysh contour. Note that while G is a contour ordered object, F and ρ are *not* contour ordered any more. A similar decomposition is possible for the self-energy. First one decomposes it into a local and a non-local part as $\Sigma(x,y) = -i\Sigma^{(0)}(x)\delta(x-y) + \overline{\Sigma}(x,y)$. Then one can write

$$\overline{\Sigma}(x,y) = \Sigma_F(x,y) - \frac{1}{2}\Sigma_\rho(x,y)\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0). \tag{2.20}$$

From the quantum equations of motion (2.15) and (2.16) one can derive evolution equations for F and ρ, which read

$$[\Box_x + M^2(x)]F(x,y) = -\int_{t_0}^{x^0} dz \Sigma_\rho(x,z)F(z,y) + \int_{t_0}^{y^0} dz \Sigma_F(x,z)\rho(z,y) \tag{2.21}$$

$$[\Box_x + M^2(x)]\rho(x,y) = -\int_{y^0}^{x^0} dz \Sigma_\rho(x,z)\rho(z,y) \tag{2.22}$$

with $\int_{t_1}^{t_2} dz \equiv \int_{t_1}^{t_2} dz^0 \int d^d z$ and $M^2(x) \equiv m^2 + \Sigma^{(0)}(x)$. Given the self-energy in some approximation, these equations can be implemented on a computer (cf. e. g. [18]), though the numerical study of non-equilibrium evolution equations is tempting due to the "memory integrals" involved in it. An alternative approach is outlined in the next chapter, consisting of an expansion of these equations in the gradients of the "central coordinates" $X \equiv (x+y)/2$. This way some analytical insight can be gained.
To close this chapter, we will briefly discuss one very important approximation for the calculation of the self-energy in the 2PI formalism, the $1/N$ or "ring diagram" expansion. Say we have an $O(N)$ symmetric scalar field theory, defined by the action

$$S[\phi_a] = \int_{x,\mathcal{C}} \left\{ \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{m^2}{2}\phi_a\phi_a - \frac{\lambda}{4!N}(\phi_a\phi_a)^2 \right\} \tag{2.23}$$

where summation over repeated indices is implied. Then it is possible to show that in an expansion in $1/N$, the next-to-leading-order contribution is given by

$$\frac{i}{2}\mathrm{Tr}\ln\left\{ \delta(x-y) + i\frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y) \right\} \tag{2.24}$$

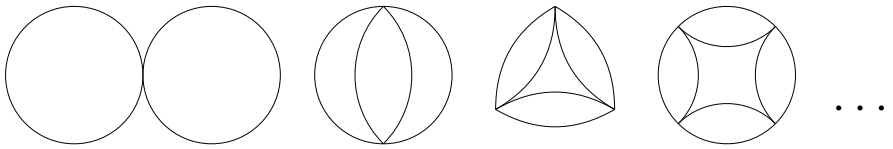

Figure 2.2: Ring diagrams giving the next-to-leading order contribution to the effective action in a $1/N$ expansion.

where the ln has to be understood as the matrix logarithm in the functional sense, i. e. taking the expression in brackets to be a matrix with indices x and y. The diagrams corresponding to this expression are all of the "ring" type as depicted in figure (2.2). From this one can calculate the self energy. Namely, one obtains

$$\overline{\Sigma}_{ab}(x,y) = -\frac{\lambda}{3N} G_{ab}(x,y) I(x,y). \tag{2.25}$$

The function $I(x,y)$ is defined by

$$iI(x,y) = \delta(x-y) - \left\{ \delta(x-y) + i\frac{\lambda}{6N} G_{ab}(x,y) G_{ab}(x,y) \right\}^{-1} \tag{2.26}$$

where the inverse is again understood in the functional sense. Multiplying with $\delta(x-y) + i\lambda/(6N) G_{ab}(x,y) G_{ab}(x,y)$ on both sides (in the matrix sense) and writing the functional matrix multiplications as integrals, one obtains an integral equation for $I(x,y)$:

$$I(x,y) = \frac{\lambda}{6N} G_{ab}(x,y) G_{ab}(x,y) - i\frac{\lambda}{6N} \int_z I(x,z) G_{ab}(z,y) G_{ab}(z,y). \tag{2.27}$$

From now on let us assume that $G_{ab}(x,y) = \delta_{ab} G(x,y)$, $F_{ab}(x,y) = \delta_{ab} F(x,y)$ and $\rho_{ab}(x,y) = \delta_{ab}\rho(x,y)$. $I(x,y)$ can again be decomposed into statistical and spectral components, $I(x,y) = I_F(x,y) - \frac{i}{2} I_\rho(x,y) \mathrm{sgn}_{\mathcal{C}}(x^0 - y^0)$. One finds that the I_F and I_ρ functions fulfill the following integral equations:

$$I_F(x,y) = \Pi_F(x,y) - \int_{t_0}^{x^0} dz I_\rho(x,z) \Pi_F(z,y) + \int_{t_0}^{y^0} dz I_F(x,z) \Pi_\rho(z,y) \tag{2.28}$$

$$I_\rho(x,y) = \Pi_\rho(x,y) - \int_{y^0}^{x^0} dz I_\rho(x,z) \Pi_\rho(z,y) \tag{2.29}$$

where

$$\Pi_F(x,y) = \frac{\lambda}{6} \left(F(x,y)^2 - \frac{1}{4}\rho(x,y)^2 \right) \tag{2.30}$$

$$\Pi_\rho(x,y) = \frac{\lambda}{3} F(x,y)\rho(x,y). \tag{2.31}$$

Here $F(x,y)^2$ and $\rho(x,y)^2$ is *not* meant in the functional sense. For the self energies one obtains

$$\Sigma_F(x,y) = -\frac{\lambda}{3N}\left(F(x,y)I_F(x,y) - \frac{1}{4}\rho(x,y)I_\rho(x,y)\right) \tag{2.32}$$

$$\Sigma_\rho(x,y) = -\frac{\lambda}{3N}\left(F(x,y)I_\rho(x,y) + \rho(x,y)I_F(x,y)\right). \tag{2.33}$$

With these expressions for the self-energy one can now either set up a numerical computation of the evolution equations or treat them by a gradient expansion in order to obtain a Boltzmann equation, as outlined in the following section.

2.3 Gradient expansion and Boltzmann equation

If one puts a system in a state far from thermal equilibrium, one usually observes that, after a very fast evolution at the beginning, at later stages the system evolves more smoothly in time. The gradient expansion aims at giving an approximate description of the system at these later times (regarding its range of validity cf. [19]).
First we introduce central coordinates $X^\mu \equiv (x^\mu + y^\mu)/2$ and relative coordinates $s^\mu \equiv x^\mu - y^\mu$. Then one can Fourier transform quantities such as F and ρ with respect to the relative time coordinates:

$$F(X;p) \equiv \int_{-2X^0}^{2X^0} ds^0 e^{is^0p^0} \int_{-\infty}^{\infty} d^d s e^{-i\mathbf{s}\cdot\mathbf{p}} F(X+s/2, X-s/2) \tag{2.34}$$

$$\rho(X;p) \equiv -i \int_{-2X^0}^{2X^0} ds^0 e^{is^0p^0} \int_{-\infty}^{\infty} d^d s e^{-i\mathbf{s}\cdot\mathbf{p}} \rho(X+s/2, X-s/2) \tag{2.35}$$

where the $-i$ is included in order to make $\rho(X;p)$ real. Analogously one also defines the Fourier transform of the statistical and spectral components of Σ, I and Π. The time integral has to be restricted to the range from $-2X^0$ to $2X^0$ because the time evolution starts at $x^0 = y^0 = 0$. Nevertheless one commonly assumes that X^0 is so large that one can also send these integration limits to infinity. This restricts the range of validity of this approach to late times.
Now one expands the evolution equation (2.21) in powers of ∂_{X^μ}, i. e. gradients of the central coordinate. Commonly, very few terms are kept. E. g. one can show that to lowest order, the evolution equations become

$$2p^\mu \frac{\partial F(X;p)}{\partial X^\mu} = \Sigma_\rho(X;p)F(X;p) - \Sigma_F(X;p)\rho(X;p) \tag{2.36}$$

$$2p^\mu \frac{\partial \rho(X;p)}{\partial X^\mu} = 0. \tag{2.37}$$

If one further assumes spatial homogeneity, they simplify further to

$$2p^0 \frac{\partial F(X;p)}{\partial X^0} = \Sigma_\rho(X;p)F(X;p) - \Sigma_F(X;p)\rho(X;p) \tag{2.38}$$

$$2p^0 \frac{\partial \rho(X;p)}{\partial X^0} = 0. \tag{2.39}$$

This means, in particular, that at this order the spectral function becomes time independent. Furthermore, the intricate memory integrals have disappeared. This order of approximation is known as *kinetic theory*. With some effort it is possible to bring these equations into a form that resembles the well-known Boltzmann equation. First one introduces a function f by writing

$$F(X;p) \equiv \left(f(X;p) + \frac{1}{2} \right) \rho(X;p). \tag{2.40}$$

From the symmetry properties of F and ρ it follows that

$$f(X;-p) = -(f(X;p) + 1). \tag{2.41}$$

Since we have assumed spatial homogeneity, there is no dependence on the spatial components of X^μ. From now on we write $X^0 \equiv t$. Now we define an effective number distribution $f(t;\mathbf{p})$ by

$$f(t;\mathbf{p}) \equiv \int_0^\infty \frac{dp^0}{2\pi} 2p^0 \rho(p) f(t;p). \tag{2.42}$$

This is motivated by the fact that if $\rho(p)$ were the free spectral function $\rho_{\text{free}}(p) = 2\pi \,\text{sgn}(p^0)\, \delta((p^0)^2 - \omega_{\mathbf{p}}^2)$ with $\omega_{\mathbf{p}} \equiv \sqrt{m^2 + \mathbf{p}^2}$ this definition would yield $f(t;\mathbf{p}) = f(t;p^0 = \omega_{\mathbf{p}}, \mathbf{p})$. With this one can rewrite (2.38) as

$$\frac{\partial f(t;\mathbf{p})}{\partial t} = \int_0^\infty \frac{dp^0}{2\pi} \left[\Sigma_\rho(t;p)F(t;p) - \Sigma_F(t;p)\rho(t;p) \right] \equiv C[f](t;\mathbf{p}). \tag{2.43}$$

By expressing everything in terms of f and ρ only, this can be cast into a form resembling even closer a Boltzmann equation. Here we take the self energies obtained by the $1/N$ resummation outlined in the last chapter.
First we need an additional definition. For a function such as $G(x,y)$, $\Sigma(x,y)$, $I(x,y)$ and $\Pi(x,y)$ which can be decomposed into F and ρ components, one defines its *retarded* and *advanced* version by $G_R(x,y) = \Theta(x^0 - y^0)\rho(x,y)$ and $G_A(x,y) = -\Theta(y^0 - x^0)\rho(x,y)$ and analogously for the other ones. It can be shown then that within the approximation $t_0 \to -\infty$ the implicit equation for I can be solved in momentum space by

$$I_F(t;p) = v_{\text{eff}}(t;p)\Pi_F(t;p) \tag{2.44}$$

$$I_\rho(t;p) = v_{\text{eff}}(t;p)\Pi_\rho(t;p) \tag{2.45}$$

where $v_{\text{eff}}(t;p)$ is an effective coupling accounting for the sum of the ring diagrams. It is given by

$$v_{\text{eff}}(t;p) = \frac{1}{|1+\Pi_R(t;p)|^2}. \tag{2.46}$$

With this it is finally possible to write the scattering integral $C[f](t;\mathbf{p})$ in a form close to that of a Boltzmann equation:

$$\begin{aligned} C[f](\mathbf{p}) = &-\frac{\lambda^2}{18N}\int_0^\infty \frac{dp^0}{2\pi}\int_{q_1q_2q_3}(2\pi)^{d+1}\delta(p-q_1-q_2-q_3) \\ &\times[(f(q_1)+1)(f(q_2)+1)(f(q_3)+1)f(p) - f(q_1)f(q_2)f(q_3)(f(p)+1)] \\ &\times \rho(q_1)\rho(q_2)\rho(q_3)\rho(p)v_{\text{eff}}(p-q_1) \end{aligned} \tag{2.47}$$

where for clarity we have suppressed the dependence on t.
One commonly makes additional assumptions about the scattering integral. The first is to assume that $\rho(p)$ is not only constant in time but also equal to the free spectral function, i. e. $\rho(p) \approx \rho_{\text{free}}(p) = 2\pi\text{sgn}(p^0)\delta((p^0)^2-\omega_{\mathbf{p}}^2)$. This is known as *on-shell* or *quasi-particle* approximation. Furthermore one assumes that $f \gg 1$ so that in (2.47) all terms containing less than three fs can be neglected and that $\Pi_R \gg 1$ so that $v_{\text{eff}} \approx 1/|\Pi_R|^2$. With these assumptions the Boltzmann equation becomes *scaling*, i. e. permits solutions of the form $f(t;\mathbf{p}) = (t/t_0)^\alpha f_S((t/t_0)^\beta\mathbf{p})$. Inserting this ansatz one can determine the exponents α and β (for clarity we set $t_0 = 1$):

$$\frac{\partial}{\partial t}[t^\alpha f_S(t^\beta\mathbf{p})] = C\left[t^\alpha f_S(t^\beta\mathbf{k})\right](\mathbf{p}) \tag{2.48}$$

$$\Leftrightarrow \alpha t^{\alpha-1}f_S(t^\beta\mathbf{p}) + \beta t^{\alpha+\beta-1}\mathbf{p}\cdot\nabla f_S|_{t^\beta\mathbf{p}} = C\left[t^\alpha f_S(t^\beta\mathbf{k})\right](\mathbf{p}). \tag{2.49}$$

Now sending $\mathbf{p}\to t^{-\beta}\mathbf{p}$ yields

$$t^{\alpha-1}[\alpha f_S(\mathbf{p}) + \beta\mathbf{p}\cdot\nabla f_S(\mathbf{p})] = t^\lambda C\left[f_S(\mathbf{k})\right](\mathbf{p}) \tag{2.50}$$

where $t^\lambda C[f_S(\mathbf{k})](\mathbf{p}) \equiv C[t^\alpha f_S(t^\beta\mathbf{k})](t^{-\beta}\mathbf{p})$ and the exponent λ has to be determined from the scattering integral. To make $f(t;\mathbf{p}) = (t/t_0)^\alpha f_S((t/t_0)^\beta\mathbf{p})$ a solution of the Boltzmann equation for all times t, one has to demand $\alpha - 1 = \lambda$. In the case of massless ϕ^4 theory, one obtains $\lambda = \alpha - \beta$, from which it follows that $\beta = 1$. However, if there is a mass gap (which can be generated dynamically even if $m = 0$ in the Lagrangian) the system behaves effectively non-relativistic at low momenta and we obtain $\lambda = \alpha - 2\beta$, i. e. $\beta = 1/2$. Particle number conservation yields $\alpha = d/2$ in this case. We will later see that the situation is different for the Sine-Gordon model where the Boltzmann equation also contains terms scaling with $\beta = 1/(2+2d)$.

2.4 Truncated Wigner approximation

An extremely important tool for the study of many-body systems out of equilibrium apart from 2PI techniques is the truncated Wigner approximation [20] (for a direct comparison of 2PI and truncated Wigner simulations see [21]). It is applicable to systems with high occupation numbers, which makes it very suitable for studying non-thermal fixed points. One can see the truncated Wigner approximation as the lowest-order term in an

expansion of the classicality of the system, e. g. characterized by the occupation numbers. Here we will only consider its non-relativistic version because our numerical calculations were mainly concerned with non-relativistic systems. Be given an operator $\Omega(\{a_i\}, \{a_i^\dagger\})$ composed of the creation and annihilation operators for the different modes of the system numbered by i. Then we define its *Wigner transform* by

$$\mathcal{W}\left\{\Omega(\{a_i\}, \{a_i^\dagger\})\right\}(\boldsymbol{\psi}, \boldsymbol{\psi}^*) \equiv \frac{1}{2^m} \int d\boldsymbol{\eta} d\boldsymbol{\eta}^* \left\langle \boldsymbol{\psi} - \frac{\boldsymbol{\eta}}{2} \right| \Omega(\{a_i\}, \{a_i^\dagger\}) \left| \boldsymbol{\psi} + \frac{\boldsymbol{\eta}}{2} \right\rangle \times \exp\left(-|\boldsymbol{\psi}|^2 - \frac{1}{4}|\boldsymbol{\eta}|^2 + \frac{1}{2}(\boldsymbol{\eta}^* \cdot \boldsymbol{\psi} - \boldsymbol{\eta} \cdot \boldsymbol{\psi}^*) \right). \quad (2.51)$$

Here m denotes the number of modes in the system, $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ are m-dimensional complex vectors, $d\boldsymbol{\eta} d\boldsymbol{\eta}^* \equiv \prod_i (d\Re\eta_i d\Im\eta_i/\pi)$ and for a general m-dimensional complex vector $\boldsymbol{\alpha}$ we define $|\boldsymbol{\alpha}\rangle \equiv \prod_i |\alpha_i\rangle_i$ with $|\alpha\rangle_i$ the coherent state of mode i with parameter α. Note that here we use a convention where the coherent states are *not* normalized, i. e. $\langle\alpha|\beta\rangle = \exp(\alpha^*\beta)$.
One can show that if Ω is symmetrically ordered, the Wigner transform can be easily obtained by substituting $a_i \to \psi_i$ and $a_i^\dagger \to \psi_i^*$. E. g.

$$a^\dagger a = \frac{1}{2}(a^\dagger a + a a^\dagger) - \frac{1}{2} \quad (2.52)$$

so that $a^\dagger a \to (\psi^*\psi + \psi\psi^*)/2 - 1/2 = |\psi|^2 - 1/2$.
Furthermore one defines the *Wigner function* as the Wigner transform of the density matrix of a system, i. e.

$$W(\boldsymbol{\psi}, \boldsymbol{\psi}^*) \equiv \mathcal{W}\{\rho\}(\boldsymbol{\psi}, \boldsymbol{\psi}^*). \quad (2.53)$$

The truncated Wigner approximation can now be defined as follows: say we have some observable $\mathcal{O}$ whose expectation value we want to calculate at time t. Then this expectation value is approximately given by

$$\langle\mathcal{O}\rangle(t) \approx \int d\boldsymbol{\psi}_0 d\boldsymbol{\psi}_0^* W_0(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0^*) \mathcal{W}\{\mathcal{O}\}(\boldsymbol{\psi}(t; \boldsymbol{\psi}_0), \boldsymbol{\psi}^*(t; \boldsymbol{\psi}_0)) \quad (2.54)$$

where W_0 is the Wigner function at $t = 0$ and $\boldsymbol{\psi}(t; \boldsymbol{\psi}_0)$ is the field that arises if the initial condition $\boldsymbol{\psi}_0$ is evolved with the classical equation of motion to time t. The condition for the validity of this approximation is that the system is somehow in a nearly classical regime where quantum fluctuations are small and can be treated perturbatively. This is e. g. the case if all modes relevant for the dynamics are highly occupied.
The truncated Wigner approximation is very suitable to be implemented on a computer. The integral over the initial conditions in (2.54) can be evaluated in a Monte Carlo fashion if the Wigner function is positive definite (which is the case if we choose the initial state to be a product of coherent states). Then all what has to be done is to evolve the system with the classical equation of motion for a high number of initial conditions chosen randomly according to the weight of the Wigner function and to average the observables over these trajectories.

3 The problem of phase correlators

Be given a Gross-Pitaevskii type quantum field theory with a complex field $\psi(\mathbf{x}, t)$. Then we can write ψ in the density-phase-representation

$$\psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} e^{i\theta(\mathbf{x},t)}. \tag{3.1}$$

The object we want to consider in this thesis are equal-time correlators of the phases $\theta(\mathbf{x}, t)$, i. e.

$$\langle \theta(\mathbf{x}, t)\theta(\mathbf{y}, t)\rangle \tag{3.2}$$

or in momentum space

$$\langle \theta(\mathbf{k}, t)\theta(-\mathbf{k}, t)\rangle. \tag{3.3}$$

Although it is easy to formally write down these expressions, their precise definition is not obvious. It turns out that there is in fact more than one possible definition. Which one of these is the "natural" one is not a priori clear.
Let us first consider the problem of defining $\langle \theta(\mathbf{x}, t)\rangle$. In the path integral formalism, the definition is straightforward:

$$\langle \theta(\mathbf{x}, t)\rangle \equiv \int D\psi D\psi^* \arg(\psi(\mathbf{x}, t)) e^{iS[\psi,\psi^*]} \tag{3.4}$$

with $\int D\psi D\psi^*$ representing a zero-temperature or non-equilibrium path integral. Translating this into the operator formalism, i. e. defining a "phase operator" $\hat{\theta}$, is a highly intricate problem. It should be defined on the entire Fock space and should translate to $\arg(\psi)$ in the path integral formalism, i. e. have the property $\langle\alpha|\hat{\theta}|\alpha\rangle = \arg(\alpha)$ for all coherent states $|\alpha\rangle$. A reasonable choice seems to be

$$\hat{\theta} \equiv \frac{1}{2i}(\ln(a) - \ln(a^\dagger)). \tag{3.5}$$

By replacing $a \to \psi$ and $a^\dagger \to \psi^*$ one would get

$$\hat{\theta} \to \frac{1}{2i}(\ln(\psi) - \ln(\psi^*)) = \arg(\psi). \tag{3.6}$$

However, one still has the problem of defining the logarithm of the creation and annihilation operators. The series expansion of $\ln(a)$ and $\ln(a^\dagger)$ converges only on a subset of all possible states, so it is necessary to somehow extend the definition. Naively one would do this by inserting a $\mathbf{1}$ in terms of the coherent states in front of $\ln(a)$, pulling $\ln(a)$ into the integral, applying it to the coherent state and defining $\ln(a)|\alpha\rangle = \ln(\alpha)|\alpha\rangle$. I. e. one would arrive at the definition

$$\ln(a) \equiv \int d\alpha d\alpha^* e^{-|\alpha|^2} \ln(\alpha)|\alpha\rangle\langle\alpha| \tag{3.7}$$

and analogously for $\ln(a^\dagger)$. While this is a perfectly well-defined definition, it is pointed out in [22] that it has a severe problem: the equation $\ln(a)|\alpha\rangle = \ln(\alpha)|\alpha\rangle$ would not be valid any more and therefore $\langle\alpha|\hat{\theta}|\alpha\rangle \neq \arg(\alpha)$, which can be seen by simply inserting

some $|\alpha\rangle$. This is somehow counterintuitive: one would think that we have not done anything but inserting a $\mathbf{1}$. The mathematical reason is that we have naively pulled the operator $\ln(a)$ into the $\int d\alpha d\alpha^*$ integral which is not allowed here because the logarithm is no smooth function in the entire complex plane but has a discontinuity at its branch cut.
As in many cases, these subtleties can be traced back to the overcompleteness of the coherent states. Therefore in that same paper the authors try to cure the described difficulties by using a complete (instead of overcomplete) subset of the coherent states, a construction due to von Neumann. However, their $\hat{\theta}$ then fulfills the condition $\langle\alpha|\hat{\theta}|\alpha\rangle = \arg(\alpha)$ only approximately.
An alternative approach is proposed in [23]. Instead of considering the Taylor series expansion of the logarithm, the author considers the Runge expansion of the logarithm. Loosely speaking, the Runge theorem states that under certain conditions an analytic function can be arbitrarily well approximated by a sequence of polynomials. Therefore a sequence of logarithm Runge polynomials in the creation and annihilation operators on a coherent state converges to the logarithm of the complex parameter of that state. However, this does not make sure yet that the resulting operator converges on the entire Fock space.
In the following, we will leave such difficulties apart and work in the c-number formulations of quantum theory, i. e. the path integral and Wigner formulation. To close this discussion, let us briefly consider the translation of $\arg(\psi) = (\ln(\psi) - \ln(\psi^*))/2i$ into Wigner space. The Wigner transform of this expression is given by

$$\frac{1}{2}\int d\eta d\eta^* \frac{1}{2i}(\ln(\psi + \eta/2) - \ln(\psi^* - \eta^*/2))e^{-|\eta|^2/2}. \tag{3.8}$$

This integral looks hardly analytically solvable. However, one can see easily that for $|\psi| \gg 1$ it will yield approximately $\arg(\psi)$ so that, as usually in this limit, the distinction between the path integral and Wigner version of an observable does not matter.
Now let us turn to phase correlators. At the first glance it is straightforward to define

$$\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle \equiv \int D\psi D\psi^* \arg(\psi(\mathbf{x},t))\arg(\psi(\mathbf{y},t))e^{iS[\psi,\psi^*]} \tag{3.9}$$

Although this definition seems straightforward, it has a problem: θ being a compact variable taking only values between $-\pi$ and π, we will have unphysical jumps in θ when ψ crosses the negative real axis. One possibility is to simply accept this fact: we will call this the "obvious" definition of the phase correlator. But there are other ways to cope with this problem. We summarize here the main options for the definition:

- Obvious definition: take the phases as they are. Unphysical phase jumps are accepted.
- Continuation definition: continue the phases onto the next "Riemann sheet" if ψ crosses the negative real axis. In finite size systems, there is some ambiguity in how to treat the boundaries in this definition.
- Modification definitions: instead of considering the product $\theta(\mathbf{x},t)\theta(\mathbf{y},t)$ consider some other function $f(\theta(\mathbf{x},t),\theta(\mathbf{y},t))$ that somehow resembles $\theta(\mathbf{x},t)\theta(\mathbf{y},t)$ but avoids the phase jumps. Of course this is no "real" phase correlator then. But

by a suitable choice of f it can be possible to capture the same physics as the phase correlator and to avoid phase jumps at the same time.

In some sense the continuation definition is the most natural one and it is the only one accessible to analytic treatment in an interacting theory. We will therefore mainly consider this definition in this thesis. It has, however, the drawback that in more than one-dimensional systems it forces us to give up the notion of one unique phase correlator. Instead it turns out that one has to consider two different objects: a "sound-wave" phase correlator and a "vortex" phase correlator. Discussions of the properties of the obvious and modification definition can mainly be found in chapter 4.
In the following two sections we have a closer look at the continuation and modification definition.

3.1 Continuation definition

For phase fields θ, which are mappings from $\mathbb{R}^n$ to $[-\pi, \pi)$, the definition of the derivative has to be generalized. Considering θ as a function in the usual sense, it is discontinuous where it jumps from $-\pi$ to π and therefore not differentiable in these points. On the other hand it is obvious that one can assign a meaningful value to the derivative of a phase also in the points where it jumps by 2π since a simple redefinition of the phase would move the jumps to other positions.
Be given a phase field $\theta : \mathbb{R}^n \to [-\pi, \pi)$, then we define the *continued derivative* $\tilde{\partial}_i\theta$ as

$$\tilde{\partial}_i\theta(\mathbf{x}) \equiv \lim_{h\to 0} \frac{\mathrm{unwrap}(\theta(\mathbf{x} + h\mathbf{e}_i) - \theta(\mathbf{x}))}{h} \tag{3.10}$$

where the unwrap function is defined as

$$\mathrm{unwrap}(x) \equiv \begin{cases} x,\ -\pi \le x \le \pi \\ x - 2\pi,\ x > \pi \\ x + 2\pi,\ x < -\pi \end{cases} . \tag{3.11}$$

An equivalent definition is

$$\tilde{\partial}_i\theta(\mathbf{x}) \equiv -ie^{-i\theta(\mathbf{x})}\partial_i e^{i\theta(\mathbf{x})}. \tag{3.12}$$

The continued gradient $\widetilde{\nabla}\theta$ and other differential operators are defined correspondingly.
With the help of the continued derivative it is possible to express the velocity field of a Gross-Pitaevskii system in terms of the phases. It is given by

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \frac{1}{|\psi|^2}\frac{1}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \frac{\widetilde{\nabla}\theta}{m}. \tag{3.13}$$

For convenience let us also define $\mathbf{V} \equiv m\mathbf{v} = \widetilde{\nabla}\theta$.
The vector field $\mathbf{V}$ is a physical field. The operator corresponding to (3.13) is hermitian and contains only pairings of creation and annihilation operators. Therefore $\mathbf{V}$ is in principle measurable.
The relation $\mathbf{V} = \widetilde{\nabla}\theta$ suggests that θ is the "potential" of $\mathbf{V}$. However, it is no real potential because $\mathbf{V}$ is given by the continued gradient of θ instead of the gradient. It would therefore be convenient to introduce a (jump-free) $\theta_{\mathrm{continued}}$ for which in fact

$\mathbf{V} = \nabla\theta_{\text{continued}}$. In one dimension $\mathbf{V}$ reduces to a simple one-dimensional function $V(x)$ and it is easy to define a unique potential $\theta_{\text{continued}}$ of $V(x)$ (simply by taking the integral over $V(x)$). However, in two and three dimensions, this is only possible if $\mathbf{V}$ is curl free. In real systems this is usually not true because quantum vortices are present. Then it becomes necessary to introduce a scalar and a vector potential of $\mathbf{V}$. We will call them the sound-wave and vortex part of the (continued) phase and denote them by θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$, respectively. I. e. we write the velocity field as

$$\mathbf{V} = \nabla\theta_{\text{sw}} + \nabla \times \boldsymbol{\theta}_{\text{vo}}. \tag{3.14}$$

We can now calculate correlators of these continued θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ fields which are not spoilt by phase jumps. The main advantage of this (apart from the fact that unnatural jumps are avoided) is that correlators of θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ are accessible to analytic treatment: writing the action in terms of these fields only ordinary differential operators will appear. In contrast, if one writes the action in terms of θ itself, the highly intricate continued gradient $\widetilde{\nabla}$ will be present in the action which cannot be subjected to standard techniques such as Fourier transforming, calculating a Green's function and performing diagrammatic expansions or integrating out fluctuations. The main drawback is that now we do not have one single phase correlator any more (at least in more than one dimension). A frequently employed assumption is that sound-wave and vortex dynamics approximately decouple. Therefore we will not consider "mixed" correlators of θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ among each other. For the sound-wave part it is obvious to consider

$$\langle\theta_{\text{sw}}(\mathbf{x},t)\theta_{\text{sw}}(\mathbf{y},t)\rangle. \tag{3.15}$$

But for the vortex part, there are several possibilities: one could consider correlators like

$$\langle\boldsymbol{\theta}_{\text{vo}}(\mathbf{x},t)\cdot\boldsymbol{\theta}_{\text{vo}}(\mathbf{y},t)\rangle \tag{3.16}$$

or

$$\langle\boldsymbol{\theta}_{\text{vo}}(\mathbf{x},t)\times\boldsymbol{\theta}_{\text{vo}}(\mathbf{y},t)\rangle \tag{3.17}$$

or correlators of single components. However, in the case of two dimensions which we will mainly consider here, $\boldsymbol{\theta}_{\text{vo}}$ effectively becomes a scalar field so that there are no ambiguities in what to correlate.
(3.14) still does not define θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ uniquely. It is convenient to require the standard conditions that $\nabla\cdot\boldsymbol{\theta}_{\text{vo}} = 0$ and that θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ decay to 0 at infinity. Then one calculate them from $\mathbf{V}$ by the Helmholtz decomposition theorem:

$$\theta_{\text{sw}}(\mathbf{r}) = -\frac{1}{4\pi}\int d^3r'\frac{\nabla\cdot\mathbf{V}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = -\frac{1}{4\pi}\int d^3r'\frac{\nabla\cdot\widetilde{\nabla}\theta(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.18}$$

$$\boldsymbol{\theta}_{\text{vo}}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r'\frac{\nabla\times\mathbf{V}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi}\int d^3r'\frac{\nabla\times\widetilde{\nabla}\theta(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.19}$$

For the two-dimensional case $\boldsymbol{\theta}_{\text{vo}}$ is a scalar θ_{vo} which can be thought of as the z-component of $\boldsymbol{\theta}_{\text{vo}}$. Here we have

$$\begin{aligned}\theta_{\text{sw}}(\mathbf{r}) &= \frac{1}{2\pi}\int d^2r'\nabla\cdot\mathbf{V}(\mathbf{r}')\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right)\\ &= \frac{1}{2\pi}\int d^2r'\nabla\cdot\widetilde{\nabla}\theta(\mathbf{r}')\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right) \qquad (3.20)\\ \theta_{\text{vo}}(\mathbf{r}) &= -\frac{1}{2\pi}\int d^2r'(\partial_x V_y(\mathbf{r}')-\partial_y V_x(\mathbf{r}'))\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right)\\ &= -\frac{1}{2\pi}\int d^2r'(\partial_x\tilde{\partial}_y\theta(\mathbf{r}')-\partial_y\tilde{\partial}_x\theta(\mathbf{r}'))\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right) \qquad (3.21)\end{aligned}$$

where a is some integration constant needed to make the argument of the logarithm dimensionless.
So far our discussion was always based on the assumption of the thermodynamic limit $V\to\infty$, i. e. that all fields live on the entire $\mathbb{R}^n$. However, numerical calculations have to be performed in a box of finite size. A frequent choice for the boundary conditions (which we will impose throughout this thesis) are periodic boundary conditions. This gives rise to the question how to define potentials in a periodic box. $\mathbf{V}$ will of course be periodic since it is directly calculated from the physical field ψ. But when calculating a potential of $\mathbf{V}$ it will typically break the periodicity (the simplest example is a constant $\mathbf{V}$: the resulting scalar potential will constantly increase throughout the box so that it cannot have the same value on the boundaries). As for the choice of boundary conditions itself, up to a certain extent it is a matter of taste how to resolve this problem. For sufficiently large boxes, i. e. approaching the thermodynamic limit, the results should be independent of such choices.
Here we will use a definition of θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$ on finite size systems that preserves periodicity but slightly modifies (3.14). To motivate our definition, we will go over from position space to momentum space. So let us first consider the momentum space version of the Helmholtz decomposition (in the entire $\mathbb{R}^3$). We define the Fourier transform of $\mathbf{V}(\mathbf{r})$ by

$$\mathbf{V}(\mathbf{k}) = \int d^3r e^{i\mathbf{r}\cdot\mathbf{k}}\mathbf{V}(\mathbf{r}). \qquad (3.22)$$

It is straightforward to show by means of standard vector identities that the following decomposition of $\mathbf{V}(\mathbf{k})$ holds:

$$\mathbf{V}(\mathbf{k}) = -i\mathbf{k}\left(i\frac{\mathbf{k}}{|\mathbf{k}|^2}\cdot\mathbf{V}(\mathbf{k})\right) - i\mathbf{k}\times\left(-i\frac{\mathbf{k}}{|\mathbf{k}|^2}\times\mathbf{V}(\mathbf{k})\right). \qquad (3.23)$$

The terms in big brackets can now be interpreted as the momentum space version of θ_{sw} and $\boldsymbol{\theta}_{\text{vo}}$:

$$\theta_{\text{sw}}(\mathbf{k}) = i\frac{\mathbf{k}}{|\mathbf{k}|^2}\cdot\mathbf{V}(\mathbf{k}) \qquad (3.24)$$

$$\boldsymbol{\theta}_{\text{vo}}(\mathbf{k}) = -i\frac{\mathbf{k}}{|\mathbf{k}|^2}\times\mathbf{V}(\mathbf{k}). \qquad (3.25)$$

	Sound Wave	Vortex
1D	$\theta_{\mathrm{sw}}(k) = \frac{i}{k}\int dx e^{ixk}\tilde{\partial}_x\theta(x)$	–
2D	$\theta_{\mathrm{sw}}(\mathbf{k}) = \frac{i\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\cdot\int d^2r e^{i\mathbf{r}\cdot\mathbf{k}}\widetilde{\nabla}\theta(\mathbf{r})$	$\theta_{\mathrm{vo}}(\mathbf{k}) = \int d^2r e^{i\mathbf{r}\cdot\mathbf{k}}\frac{-i(k_x\tilde{\partial}_y\theta(\mathbf{r}) - k_y\tilde{\partial}_x\theta(\mathbf{r}))}{\lvert\mathbf{k}\rvert^2}$
3D	$\theta_{\mathrm{sw}}(\mathbf{k}) = \frac{i\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\cdot\int d^3r e^{i\mathbf{r}\cdot\mathbf{k}}\widetilde{\nabla}\theta(\mathbf{r})$	$\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{k}) = \frac{-i\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\times\int d^3r e^{i\mathbf{r}\cdot\mathbf{k}}\widetilde{\nabla}\theta(\mathbf{r})$

Table 1: Definition of sound-wave and vortex part of the phase in different dimensions.

That this is justified can be seen by going back to real space:

$$\begin{aligned}
&\nabla\theta_{\mathrm{sw}}(\mathbf{r}) + \nabla\times\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{r}) = \nabla\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}\theta_{\mathrm{sw}}(\mathbf{k}) + \nabla\times\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{k})\\
&= \nabla\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}\left(i\frac{\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\cdot\mathbf{V}(\mathbf{k})\right) + \nabla\times\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}\left(-i\frac{\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\times\mathbf{V}(\mathbf{k})\right)\\
&= \int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}(-i\mathbf{k})\left(i\frac{\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\cdot\mathbf{V}(\mathbf{k})\right) + \int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}(-i\mathbf{k})\times\left(-i\frac{\mathbf{k}}{\lvert\mathbf{k}\rvert^2}\times\mathbf{V}(\mathbf{k})\right)\\
&= \int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{V}(\mathbf{k}) = \mathbf{V}(\mathbf{r}).
\end{aligned} \tag{3.26}$$

These considerations still are only valid for functions living on $\mathbb{R}^3$. In a periodic box the situation is different. Here the momenta are quantized and the momentum integrals become momentum sums. Nevertheless, we will now take (3.24) and (3.25) as *defining equations* of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$ on a periodic box. (3.24) and (3.25) can be straightforwardly modified for the lower-dimensional cases. In 2D, the vortex part of the phase reduces to a scalar field again whereas in 1D the vortex part is not even present, i. e. we have one unique phase. In table (1) the definitions are summarized for all three dimensions. For the case $\mathbf{k} = 0$ these definitions cannot be applied. This reflects the fact that the global average of the phase is arbitrary. A natural choice is to set $\theta_{\mathrm{sw}}(\mathbf{k} = 0) = 0$ and $\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{k} = 0) = 0$.

By defining the phases by a momentum space expression, we make sure that periodicity is preserved: applying the discrete Fourier transform on a periodic box to $\theta_{\mathrm{sw}}(\mathbf{k})$ and $\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{k})$ we get a $\theta_{\mathrm{sw}}(\mathbf{r})$ and $\boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{r})$ which is periodic. On the other hand, as was pointed out above, this means that (3.14) cannot be fulfilled exactly any more. Let us have a short look on what this means in detail. For simplicity we restrict ourselves to the one-dimensional case. So be given a velocity field $V(x) : [0, L) \to \mathbb{R}$ which is continuous and periodic on $[0, L)$. We want to compare the definition of θ_{sw} as given in table (1) and a definition that

preserves $\partial_x \theta_{\rm sw} = V(x)$ exactly. The latter is given by

$$\int_0^x dx' V(x') + \text{const.} \tag{3.27}$$

Now let us transform the former into position space in order to compare:

$$\begin{aligned}
&\frac{1}{L}\sum_{k\neq 0} e^{-ikx}\frac{i}{k}\int_0^L dx' e^{ikx'}V(x') = \frac{1}{L}\sum_{k\neq 0} e^{-ikx}\frac{i}{k}\int_0^L dx' e^{ikx'}\partial_{x'}\int_0^{x'} dx'' V(x'') \\
&= \frac{1}{L}\sum_{k\neq 0} e^{-ikx}\int_0^L dx' e^{ikx'}\int_0^{x'} dx'' V(x'') + \frac{1}{L}\sum_{k\neq 0} e^{-ikx}\frac{i}{k}e^{ikL}\int_0^L dx' V(x') \\
&= \int_0^x dx' V(x') - \frac{1}{L}\int_0^L dx'\int_0^{x'} dx'' V(x'') - \frac{x-L/2}{L}\int_0^L dx' V(x')
\end{aligned} \tag{3.28}$$

where in the second line we have performed a partial integration and in the third line made use of $1/L\sum_{k\neq 0} e^{-ikx} = \delta(x) - 1/L$ and of the fact that in the second term we have the Fourier series of the saw tooth function. This permits now a simple interpretation. The first and second term represent a definition of $\theta_{\rm sw}(x)$ which preserves $\partial_x \theta_{\rm sw}(x) = V(x)$ exactly (the second term is just a constant that makes sure that the average of $\theta_{\rm sw}(x)$ is 0, a consequence of our choice to set the 0-momentum-mode to 0). The third term, however, subtracts a saw tooth function that exactly compensates the jump between 0 and L and therefore enforces periodicity.
Of course one could also argue that the exact fulfillment of $\partial_x \theta_{\rm sw}(x) = V(x)$ should be primary instead of spatial periodicity. As mentioned above, up to a certain extent this is a matter of taste since for the thermodynamic limit the definitions should agree. Here we have taken the momentum space expression (and therefore periodicity) to be primary for two reasons. The first one is that the analytical theory is conveniently formulated in momentum space (instead of position space). Say one term of the action is of the kind

$$\int_{\mathcal{V}} d^3 r \widetilde{\nabla}\theta \cdot \widetilde{\nabla}\theta \tag{3.29}$$

with a periodic box of volume $\mathcal{V}$ and $\widetilde{\nabla}\theta$ is curl free. Then one can rewrite this term in momentum space as

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{k}} \mathbf{k}^2 \theta_{\rm sw}(\mathbf{k})\theta_{\rm sw}(-\mathbf{k}) \tag{3.30}$$

if and only if one has defined $\theta_{\rm sw}(\mathbf{k})$ as in table (1). The second reason is numerical convenience. Calculating $\theta_{\rm sw}$ this way is quite simple: calculate $\widetilde{\nabla}\theta$, Fourier transform the result by means of a fast Fourier transform (FFT) algorithm and apply the formulas of table (1). Especially in higher dimensions taking position space to be primary would be quite unpractical.
In the following, the definition of the phase correlator as outlined in this section will be

the main working base for numerical calculations.

3.2 Modification definitions

An alternative to $\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle$ frequently employed in the literature is

$$\langle e^{i\theta(\mathbf{x},t)}e^{-i\theta(\mathbf{y},t)}\rangle. \tag{3.31}$$

Obviously, this is not spoilt by phase jumps. The motivation to relate this object to the phase correlator is that for small phase fluctuations we can expand $e^{i\theta(\mathbf{x},t)} \approx 1 + i\theta(\mathbf{x},t)$ and then ignoring (irrelevant) constants and expectation values of the fields, we get $\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle$.
Here we want to propose a second alternative that is in some sense closer to $\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle$ but has its own drawbacks. To motivate it, consider the relation

$$\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle = -\frac{1}{2}\langle(\theta(\mathbf{x},t)-\theta(\mathbf{y},t))^2\rangle + \frac{1}{2}\langle\theta(\mathbf{x},t)^2\rangle + \frac{1}{2}\langle\theta(\mathbf{y},t)^2\rangle. \tag{3.32}$$

The second and third term on the right hand side can assumed to be constants, i. e. the behavior of $\langle\theta(\mathbf{x},t)\theta(\mathbf{y},t)\rangle$ is captured by $-\langle(\theta(\mathbf{x},t)-\theta(\mathbf{y},t))^2\rangle/2$. This is an expression depending solely on the difference between the phases at two different positions. Now we can get rid of the phase jumps by defining the difference of two phases by the "shorter" distance on the unit circle. This leads to the correlator

$$-\frac{1}{2}\langle\mathrm{mindist}(\theta(\mathbf{x},t),\theta(\mathbf{y},t))^2\rangle + c \tag{3.33}$$

where the mindist function of two compact variables $x, y \in [-\pi,\pi)$ is defined as

$$\mathrm{mindist}(x,y) \equiv \begin{cases} |x-y|, |x-y| < \pi \\ 2\pi - |x-y|, |x-y| \geq \pi \end{cases} \tag{3.34}$$

and c is a constant. It is conveniently chosen such that the correlator is "connected", i. e. giving 0 for two phase variables which are completely uncorrelated and randomly distributed around the unit circle. It is easy to how that in such a case we have

$$\langle\mathrm{mindist}(\theta(\mathbf{x},t),\theta(\mathbf{y},t))^2\rangle = \pi^2/3 \tag{3.35}$$

so that one has to choose $c = \pi^2/6$.
Although this correlator is free of jumps, it is still not free of kinks: when the phases at point $\mathbf{x}$ and $\mathbf{y}$ cross the point of opposition on the unit circle the correlator has a kink, i. e. is non-differentiable. But at least the points where the kinks happen are determined by a somehow natural condition (crossing of the point of opposition) whereas the position of the jumps in the obvious definition depends on how one defines the phase.
A further disadvantage of this definition is that it can only be formulated naturally in position space. Once calculated as a function of the difference $\mathbf{x}-\mathbf{y}$, we can of course still Fourier transform the result. But this is not a very natural momentum space version. Apart from this question, it is also numerically quite inconvenient. Calculating a correlator in position space directly is computationally very intensive. In contrast, calculating a momentum space correlator directly (and in case Fourier transforming the result back to get the position space correlator) is numerically favorable due to the applicability of fast

Fourier transform (FFT) algorithms.
It should be stressed again that the two correlators discussed in this subsection are not entirely the same as a phase correlator. Nevertheless they are able to give some physical insights. We will, however, not devote them as much space as to the continuation definition.

4 One-dimensional spin system

Before turning to the more complicated case of higher-dimensional systems we will first consider a one-dimensional system as a toy system. This does not imply that the one-dimensional case is trivial (which in fact it is not). But here one can study in a clearer way than in higher dimensional systems the effect of the different definitions of the phase correlator.
We examined phase correlators for the 1D spin system described in [24]. The Hamiltonian is given by

$$H = \int dx \left[\vec{\Phi}^\dagger \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + q f_z^2 \right) \vec{\Phi} + \frac{c_0}{2} n^2 + \frac{c_1}{2} |\vec{F}|^2 \right] \tag{4.1}$$

with $\vec{\Phi}$ a three-component Bose field, $n = \vec{\Phi}^\dagger \vec{\Phi}$, $\vec{F} = \vec{\Phi}^\dagger \vec{f} \vec{\Phi}$, $\vec{f}$ the vector of the three-dimensional spin matrices and constants q, c_0 and c_1. For details of the initial conditions and parameters we refer the reader to the mentioned paper. The simulations have been performed by C.-M. Schmied who kindly provided the raw simulation data. Generally we took $N = 5000$ truncated-Wigner runs to calculate observables. As in this publication, we considered the complex field

$$F_\perp \equiv F_x + iF_y. \tag{4.2}$$

However, instead of considering $\langle |F_\perp(k)|^2 \rangle$ we examined correlators of the phase of $F_\perp$, i. e. $\langle \theta_{F_\perp}(k) \theta_{F_\perp}(-k) \rangle$, thereby applying the different definitions of phase correlators discussed above. We begin with the obvious definition, subsequently we discuss the modification definitions and finally the continuation definition. In the end we discuss the different results together and compare them.

4.1 Obvious definition

In figure (4.1) the momentum space correlator $\langle \theta(k)\theta(-k) \rangle$ according to the obvious definition is plotted for four different times. One observes that at small momenta a plateau is present, at intermediate momenta we have a clear power law and finally a UV-tail. Scaling in time is clearly visible. We have extracted the scaling exponents α and β according to the procedure described in [15] (as always in the following). The result is

$$\alpha = 0.26 \pm 0.02 \tag{4.3}$$
$$\beta = 0.29 \pm 0.02. \tag{4.4}$$

In appendix E a rescaled plot can be found (as for all the following correlator plots).
As will turn out later when considering the continuation definition, the plateau is caused by 2π-jumps in the phase. The mean distance between these jumps introduces a characteristic length scale into the system which corresponds to a characteristic momentum scale at which the plateau forms. It is therefore surprising that nevertheless the extracted exponents fulfill very well the particle number conservation condition $\alpha = d\beta$. Moreover, they show approximate agreement with the exponents for $\langle |F_\perp(k)|^2 \rangle$ extracted in [24]. We will comment on this below.
The momentum space correlator in the obvious definition also shows an anomalous feature. As one can clearly see in figure (4.1), the power law is extraordinary. The exponent

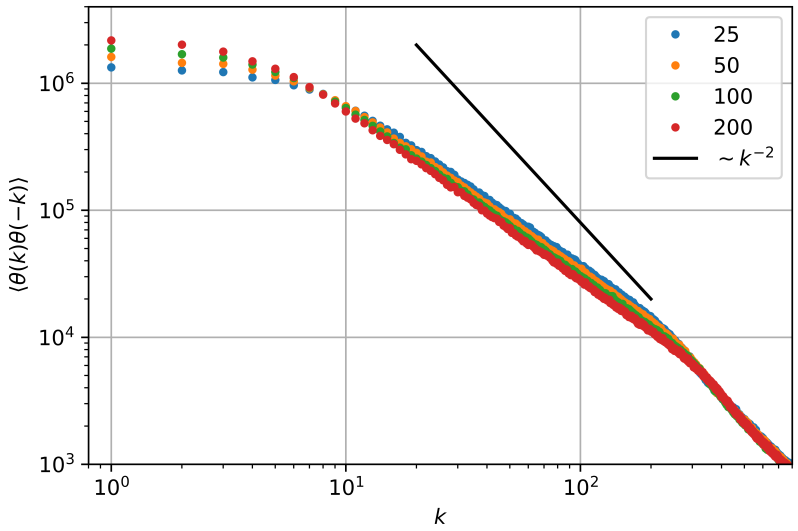

Figure 4.1: The phase correlator calculated with the obvious definition.

κ is no integer but "fractional", i. e. it lies between 1 and 2. In 1D, one would expect $\kappa = 2$ [25]. This is also the exponent that comes out for $\langle|F_\perp(k)|^2\rangle$ and will come out for the continuation definition.
We want to show that this anomalous power law is also due to the presence of the 2π-jumps in the phase. To that end consider a function $\theta(x_i) \equiv \theta_i$ defined on an infinite, one-dimensional lattice which can take only two values, π and $-\pi$. Between two neighboring lattice sites θ jumps from one value to the other with a probability p. We want to derive the correlation function

$$\langle\theta_i\theta_{i+\ell}\rangle. \tag{4.5}$$

Consider first the case that ℓ is even. The probability that θ_i and $\theta_{i+\ell}$ have the same value is given by the probability that the number of jumps between i and $i+\ell$ is even, i. e.

$$\sum_{k=0}^{\ell/2} \binom{\ell}{2k} p^{2k}(1-p)^{\ell-2k} = \frac{1}{2} + \frac{1}{2}(2p-1)^\ell. \tag{4.6}$$

Then the expectation value of $\theta_i\theta_{i+\ell}$ is given by

$$\pi^2\left(\frac{1}{2} + \frac{1}{2}(2p-1)^\ell\right) - \pi^2\left(1 - \left(\frac{1}{2} + \frac{1}{2}(2p-1)^\ell\right)\right) = \pi^2(2p-1)^\ell. \tag{4.7}$$

The case that ℓ is odd is similar. Here we have

$$\sum_{k=0}^{(\ell-1)/2} \binom{\ell}{2k} p^{2k}(1-p)^{\ell-2k} = \frac{1}{2} - \frac{1}{2}(2p-1)^\ell \tag{4.8}$$

and the expectation value of $\theta_i\theta_{i+\ell}$ is

$$\pi^2\left(\frac{1}{2}-\frac{1}{2}(2p-1)^\ell\right)-\pi^2\left(1-\left(\frac{1}{2}-\frac{1}{2}(2p-1)^\ell\right)\right)=-\pi^2(2p-1)^\ell. \tag{4.9}$$

So in total we have

$$\langle\theta_i\theta_{i+\ell}\rangle=\pi^2\begin{cases}(2p-1)^\ell, \ell\,\text{even}\\ -(2p-1)^\ell, \ell\,\text{odd}\end{cases}. \tag{4.10}$$

Now let us transform this result into momentum space, i. e. consider

$$f(k)\equiv\sum_{\ell=-\infty}^{\infty}e^{ik\ell}\langle\theta_i\theta_{i+\ell}\rangle=-\langle\theta_i^2\rangle+2\Re\sum_{\ell=0}^{\infty}e^{ik\ell}\langle\theta_i\theta_{i+\ell}\rangle. \tag{4.11}$$

In order to evaluate this sum we split it up into a sum over the even and odd ℓs and insert the formulas derived above. Thus we arrive at

$$\begin{aligned}f(k)&=-\pi^2+2\Re\sum_{m=0}^{\infty}e^{ik2m}\pi^2(2p-1)^{2m}-2\Re\sum_{m=0}^{\infty}e^{ik(2m+1)}\pi^2(2p-1)^{2m+1}\\&=\pi^2\left(2\Re\frac{1}{1-e^{2ik}(2p-1)^2}-2\Re\frac{e^{ik}(2p-1)}{1-e^{2ik}(2p-1)^2}-1\right)\\&=\pi^2\frac{1-(2p-1)^2}{1+(2p-1)^2+2(2p-1)\cos k}\end{aligned} \tag{4.12}$$

which is the desired result. Since p is the probability that a phase jump occurs between two lattice points, we can interpret $1/p$ as the mean distance between two jumps expressed in units of the lattice spacing. Then there are two ways of interpreting the dependence of (4.12) on $1/p$: either one considers the lattice spacing as fixed and thus regards $1/p$ as the (real) mean distance between two jumps, or one keeps the mean distance between two jumps fixed and considers a variation of $1/p$ as a change in lattice resolution. We will do the latter now, i. e. we introduce the momentum in units of the inverse jump length $\tilde{k}\equiv k/p$ and rewrite (4.12) as

$$\tilde{f}(\tilde{k})\equiv pf(p\tilde{k})=p\pi^2\frac{1-(2p-1)^2}{1+(2p-1)^2+2(2p-1)\cos p\tilde{k}} \tag{4.13}$$

where we have multiplied with p in order to account for the changing lattice spacing. In figure (4.2) $\tilde{f}(\tilde{k})$ is plotted as a function of $\tilde{k}$ for different values of p. One can see that indeed the resulting power law exponents are smaller than 2 but approach 2 as $p\to 0$, i. e. the lattice resolution goes to infinity. In fact it is straightforward to show that

$$\lim_{p\to 0}p\pi^2\frac{1-(2p-1)^2}{1+(2p-1)^2+2(2p-1)\cos p\tilde{k}}=\frac{4\pi^2}{4+\tilde{k}^2}. \tag{4.14}$$

This shows that the anomalous power law for the phase correlator in the obvious definition is indeed caused by the phase jumps but can also be considered as an artifact of the finite lattice resolution. We would, however, not consider this as a purely numerical artifact because in condensed matter physics every system also has a physical lattice spacing.

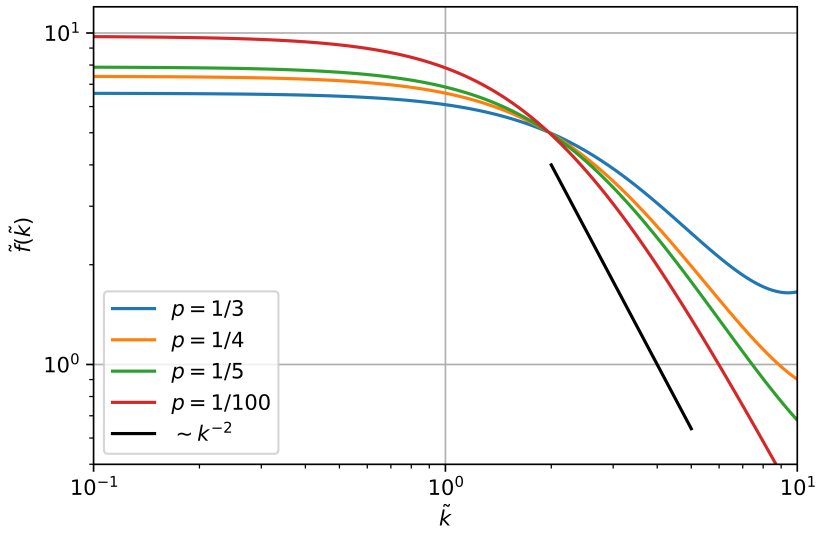

Figure 4.2: The function $\tilde{f}(\tilde{k})$.

Nevertheless this result demonstrates again the problems of the obvious definition for the phase correlator since different numerical setups can give qualitatively completely different results.

4.2 Modification definition

Figure (4.3) shows a "phase correlator" as obtained by applying (3.33) in position space, averaging over all lattice points, and then Fourier transforming the result. Apparently now the anomalous power law is not present any more but we have $\kappa = 2$. The reason is that we have constructed the correlator such as to avoid jumps. A plateau, however, is still present. This is due to the presence of kinks in the correlator, which occur when the phases cross the point of opposition on the unit circle. As for the jumps in the obvious definition, the mean distance between these kinks plays the role of a characteristic length scale responsible for the observed plateau.
For the scaling exponents we obtain

$$\alpha = 0.26 \pm 0.02 \tag{4.15}$$

$$\beta = 0.27 \pm 0.01, \tag{4.16}$$

which is in excellent agreement with the scaling exponents obtained above. This shows that our modified definition, though no real phase correlator, is still capable of giving physical insight: the scaling behavior comes out the same as for the obvious definition and additionally the artificial anomalous power law is avoided.

4.3 Continuation definition

In figure (4.4) the continuation definition correlator is shown. Apparently there is no plateau any more now. This suggests that there is no characteristic length scale or at least that it is much larger than the system size. The power law exponent is $\kappa = 2$. α and β cannot be extracted independently any more now. If $f_S(k) = k^{-\kappa}$, the scaling

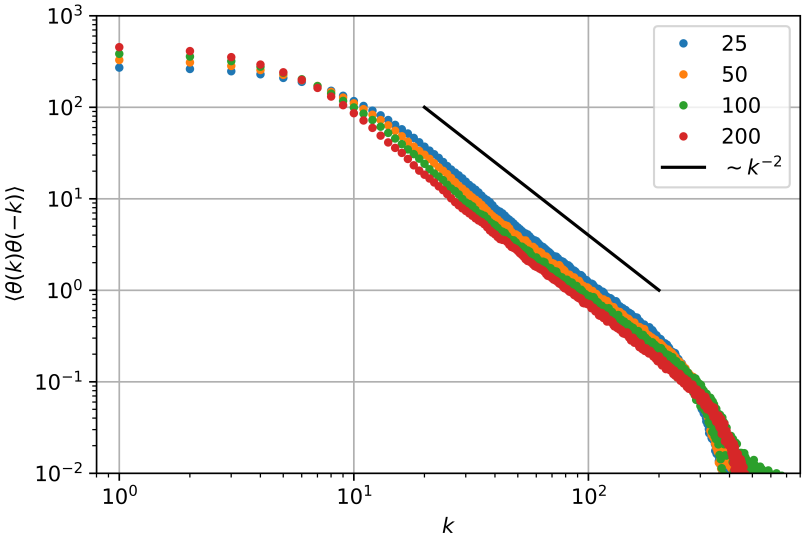

Figure 4.3: The phase correlator calculated with the modification definition.

assumption yields

$$f(t;k) = (t/t_0)^{\alpha} f_S((t/t_0)^{\beta} k) = (t/t_0)^{\alpha-\kappa\beta} k^{-\kappa} \tag{4.17}$$

so that we have only one exponent $\gamma \equiv \alpha - \kappa\beta$. The fitting procedure yields for this exponent

$$\gamma = -0.28 \pm 0.04. \tag{4.18}$$

If we *define* α and β by assuming additionally $\alpha = d\beta$ this gives $\alpha = \beta = 0.28 \pm 0.04$.

4.4 Concluding remarks about the different phase correlators in 1D

For both the obvious and modification definition we found $\alpha, \beta \sim 0.25$. For the continuation definition, which yields no plateau so that α and β are indistinguishable, the same is true if one defines $\alpha = d\beta$. Interestingly, this agrees very well with the result that in [24] was obtained for $\langle |F_\perp(k)|^2 \rangle$. This suggests that the phases are the relevant degree of freedom which drives the dynamics. Moreover, at least in terms of scaling, the different definitions of phase correlators seem to be equivalent.
Regarding power laws and plateaus, however, they are not. The obvious and modification definition both have a plateau due to the presence of jumps and kinks. Additionally the obvious definition features also an anomalous power law which can be explained by a simple model of random jumps on a lattice.
A peculiarity of phase correlators in contrast to usual correlators of the fields, which will show up again in 2D, is that they show coherence on much larger length scales. Of course the obvious and modification phase correlator showed a plateau in the IR. But this is only due to somehow unphysical artifacts, namely jumps and kinks. The continuation definition correlator is free of such artifacts and therefore lacks a plateau. Additionally to the described problems of the other definitions, this demonstrates the superiority of

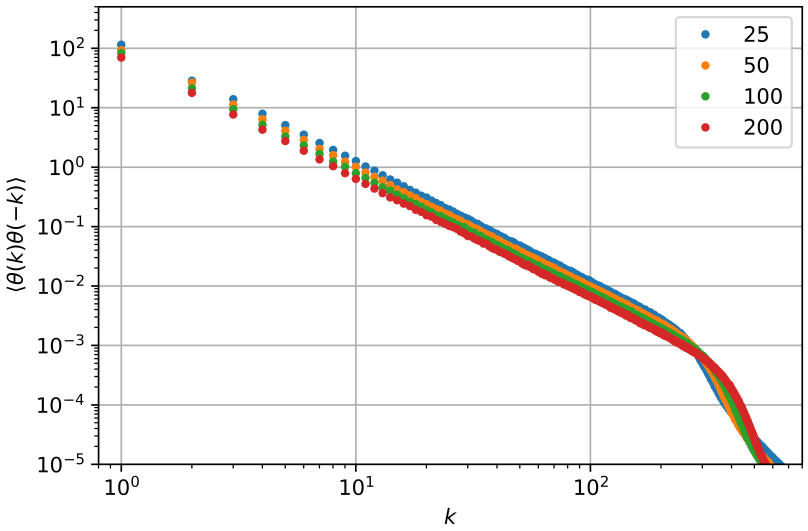

Figure 4.4: The phase correlator calculated with the continuation definition.

the continuation definition. It will therefore be the only one we will use for the higher dimensional systems.
In principle in 1D the phase can also have physical jumps due to solitons which would introduce a length scale even for the continuation definition. The fact that here no plateau appeared shows that in the studied system the "jumps" are rather smooth.
That the phases show coherence on a much larger length scale compared to correlators of the fields suggests that they are a very "clean" observable. The effects of density fluctuations are excluded. This makes it a very suitable observable for studying non-thermal fixed points.

5 Analytical considerations about phase correlators

Based on the continuation definition, we want to gain some analytical understanding of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$ in this chapter, whereby in part we review earlier findings.

5.1 Phase representation of Gross-Pitaevskii theory

Inserting the density-phase representation $\psi_a = \sqrt{\rho_a} e^{i\theta_a}$ into the $U(N)$ symmetric Gross-Pitaevskii Lagrangian (1.1) yields

$$\mathcal{L}_{\rho\theta} = -\sum_a \left\{ \rho_a \tilde{\partial}_t \theta_a + \frac{1}{2m} \left(\rho_a (\widetilde{\nabla} \theta_a)^2 + (\nabla \sqrt{\rho_a})^2 \right) \right\} - \frac{g}{2} \left(\sum_a \rho_a \right)^2 \tag{5.1}$$

where again the tilde denotes the continued derivative. The path integral over the fundamental Bose fields $\int \prod_a D\psi_a^* D\psi_a$ is transformed to $\int \prod_a D\theta_a D\rho_a \rho_a$. Now we want to get rid of the continued derivatives by a change of variables. For the gradient term this is easily done by employing $\widetilde{\nabla}\theta = \nabla\theta_{\mathrm{sw}} + \nabla \times \boldsymbol{\theta}_{\mathrm{vo}}$. However, there is also a term with a time derivative of θ. One way to treat this problem is to express $\tilde{\partial}_t \theta$ in terms of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$. Alternatively, at least in 2D, one can give up the notion of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$ and instead consider $(\tilde{\partial}_t\theta, \tilde{\partial}_x\theta, \tilde{\partial}_y\theta)$ as a three-component vector field in three-dimensional space time that can be decomposed into a divergence free and curl free part. We will first consider the former option while adopting the latter later for convenience. We will then also show that for the case of vortex-free systems and systems with sufficiently slow vortices the two points of view agree.
So let us now try to find a way to express $\tilde{\partial}_t \theta$ in terms of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$. Let us first consider the two-dimensional case, i. e we have $\widetilde{\nabla}\theta = \nabla\theta_{\mathrm{sw}} + (\partial_y \theta_{\mathrm{vo}}, -\partial_x \theta_{\mathrm{vo}})^T$. We need a way to recover the original phase θ from θ_{sw} and θ_{vo}. First one can note that the line integral of $\widetilde{\nabla}\theta$ between two points gives the difference of θ in the two points modulo an integer multiple of 2π, i. e.

$$\int_A^B d\mathbf{r} \cdot \widetilde{\nabla}\theta = \theta(B) - \theta(A) + 2\pi n, \;\; n \in \mathbb{Z}. \tag{5.2}$$

Therefore we have for a closed integration contour $\mathcal{C}$

$$\oint_{\mathcal{C}} d\mathbf{r} \cdot \widetilde{\nabla}\theta = 2\pi n, \;\; n \in \mathbb{Z}. \tag{5.3}$$

By inserting the representation of $\widetilde{\nabla}\theta$ in terms of θ_{sw} and θ_{vo} and applying the two-dimensional Stokes theorem it follows easily that the curl of $(\partial_y \theta_{\mathrm{vo}}, -\partial_x \theta_{\mathrm{vo}})^T$, i. e. $-(\partial_x^2 + \partial_y^2)\theta_{\mathrm{vo}}$, has to be of the form

$$-(\partial_x^2 + \partial_y^2)\theta_{\mathrm{vo}}(\mathbf{x}, t) = 2\pi \sum_i q_i \delta^{(2)}(\mathbf{x} - \mathbf{X}_i(t)) \tag{5.4}$$

with integer vortex charges q_i and vortex trajectories $\mathbf{X}_i(t)$. We can solve this equation for $\theta_{\mathrm{vo}}(\mathbf{x},t)$ applying standard techniques:

$$\theta_{\mathrm{vo}}(\mathbf{x},t) = -\sum_i q_i \ln\left(\frac{|\mathbf{x}-\mathbf{X}_i(t)|}{a}\right) \tag{5.5}$$

with a an integration constant. For the curl of $\theta_{\mathrm{vo}}(\mathbf{x},t)$ we obtain

$$\begin{pmatrix}\partial_y\\-\partial_x\end{pmatrix}\theta_{\mathrm{vo}}(\mathbf{x},t) = \sum_i q_i \frac{1}{(x-X_i(t))^2+(y-Y_i(t))^2}\begin{pmatrix}-(y-Y_i(t))\\x-X_i(t)\end{pmatrix} \tag{5.6}$$

where x, y and $X_i(t), Y_i(t)$ denote the components of $\mathbf{x}$ and $\mathbf{X}_i(t)$, respectively.
Now we have the tools in our hands to calculate $\tilde{\partial}_t\theta(\mathbf{x},t)$. Equation (5.2) can also be read as

$$\theta(\mathbf{x}) = \int_A^{\mathbf{x}} d\mathbf{r}\cdot\widetilde{\nabla}\theta + \theta(A) + 2\pi n. \tag{5.7}$$

If we now choose the point A to lie infinitely far away we can assume that $\theta(A)$ is not affected by the dynamics and is thus constant in time. Then we have

$$\tilde{\partial}_t\theta(\mathbf{x},t) = \tilde{\partial}_t\int_A^{\mathbf{x}} d\mathbf{r}\cdot\widetilde{\nabla}\theta = \partial_t\int_A^{\mathbf{x}} d\mathbf{r}\cdot\widetilde{\nabla}\theta \tag{5.8}$$

where in the second equality we have replaced $\tilde{\partial}_t$ by ∂_t assuming that the contour is chosen such that it does not cross vortices and that the line integral therefore is continuous. Without any loss of generality we now choose A to be $\mathbf{x}+(-\infty,0)^T$ and the contour to be a straight line between A and $\mathbf{x}$. Inserting the representation of $\widetilde{\nabla}\theta$ in terms of θ_{sw} and θ_{vo} we get

$$\begin{aligned}
\tilde{\partial}_t\theta(\mathbf{x},t) &= \partial_t\int_A^{\mathbf{x}} d\mathbf{r}\cdot\left[\nabla\theta_{\mathrm{sw}} + \begin{pmatrix}\partial_y\\-\partial_x\end{pmatrix}\theta_{\mathrm{vo}}\right]\\
&= \partial_t\theta_{\mathrm{sw}}(\mathbf{x},t)\\
&\quad + \partial_t\sum_i q_i\int_{-\infty}^{0} d\xi\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{(x+\xi-X_i(t))^2+(y-Y_i(t))^2}\begin{pmatrix}-(y-Y_i(t))\\x+\xi-X_i(t)\end{pmatrix}\\
&= \partial_t\theta_{\mathrm{sw}}(\mathbf{x},t) + \partial_t\sum_i q_i\arctan\left(\frac{y-Y_i(t)}{x-X_i(t)}\right)
\end{aligned} \tag{5.9}$$

where for the treatment of the sound-wave part we have used the fact that the line integral of a vector field which is the gradient of a potential is just the difference of the potential at the ends of the contour and that θ_{sw} is constant at infinity. The vortex part still contains a singularity, namely when $x - X_i(t) = 0$. This can be cured by applying $\arctan(x) = -\arctan(1/x) + \mathrm{const}$ (which corresponds to choosing A to lie at $\mathbf{x}+(0,-\infty)^T$ in the first place). Only if $x-X_i(t) = y-Y_i(t) = 0$ it is not well-defined any more because in the core of a vortex the phase is ill-defined. One should also note that

this representation of the phase field of a vortex is only valid "inside the time derivative". Clearly one may *not* write $\theta(\mathbf{x},t) = q \arctan\left((y-Y(t))/(x-X(t))\right)$ since the arctan-function is restricted to the interval $(-\pi/2, \pi/2)$ whereas the phase around a vortex takes every value between $-\pi$ and π at least once.
Let us now briefly discuss the generalization to the three-dimensional case. Here the condition (5.3) enforces $\nabla \times \nabla \times \boldsymbol{\theta}_{\mathrm{vo}} = -\Delta \boldsymbol{\theta}_{\mathrm{vo}}$ to be of the form

$$-\Delta \boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{x},t) = 2\pi \sum_i I_i \int d\tau \delta^{(3)}(\mathbf{x} - \mathbf{X}_i(\tau;t)) \frac{\partial \mathbf{X}_i}{\partial \tau} \tag{5.10}$$

where $\mathbf{X}_i(\tau;t)$ is a parametrization of the ith vortex tube at time t with curve parameter τ and I_i are the integer currents of the vortex tubes. To see that this expression in fact satisfies (5.3) it is sufficient to show that $\Delta \boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{x},t)$ integrated over an arbitrary surface gives an integer multiple of 2π. The Stokes theorem then ensures that the line integral of $\tilde{\nabla}\theta$ over arbitrary closed curves is an integer multiple of 2π, too. So be given a positively oriented surface $\mathbf{s}(\sigma_1, \sigma_2)$ parametrized by σ_1 and σ_2. Then we have, suppressing the dependence on i and t

$$2\pi I \int d\sigma_1 d\sigma_2 \int d\tau \left(\frac{\partial \mathbf{s}}{\partial \sigma_1} \times \frac{\partial \mathbf{s}}{\partial \sigma_2} \right) \cdot \frac{\partial \mathbf{X}}{\partial \tau} \delta^{(3)}(\mathbf{s}(\sigma_1,\sigma_2) - \mathbf{X}(\tau)) = 2\pi I \tag{5.11}$$

where we have made use of the transformation formula for the delta function

$$\delta^{(n)}(\vec{f}(\mathbf{x})) = \frac{\delta^{(n)}(\mathbf{x} - \bar{\mathbf{x}})}{|\det J[\vec{f}](\bar{\mathbf{x}})|} \tag{5.12}$$

where $\bar{\mathbf{x}}$ is the zero of $\vec{f}(\mathbf{x})$ and $J[\vec{f}]$ its Jacobian and of the fact that in three dimensions the determinant of a matrix is equal to the triple product of its column vectors. Again by applying standard techniques from (5.10) we get for the curl of $\boldsymbol{\theta}_{\mathrm{vo}}$ the Biot-Savart law

$$\nabla \times \boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{x},t) = \frac{1}{2} \sum_i I_i \int d\tau \frac{\partial \mathbf{X}_i}{\partial \tau} \times \frac{\mathbf{x} - \mathbf{X}_i(\tau;t)}{|\mathbf{x} - \mathbf{X}_i(\tau;t)|^3}. \tag{5.13}$$

$\tilde{\partial}_t \theta$ can now be calculated by the same procedure as in the 2D case. The sound-wave part gives again only $\partial_t \theta_{\mathrm{sw}}$, so we focus on the vortex part. Analogously to the 2D case we obtain:

$$\begin{aligned}
\tilde{\partial}_t \theta(\mathbf{x},t) &= \partial_t \theta_{\mathrm{sw}}(\mathbf{x},t) + \partial_t \int_A^{\mathbf{x}} d\mathbf{r} \cdot \nabla \times \boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{x},t) \\
&= \partial_t \theta_{\mathrm{sw}}(\mathbf{x},t) + \partial_t \frac{1}{2} \sum_i I_i \int_{-\infty}^{0} d\xi \int d\tau \mathbf{e}_x \cdot \left(\frac{\partial \mathbf{X}_i}{\partial \tau} \times \frac{\mathbf{x} + \xi \mathbf{e}_x - \mathbf{X}_i(\tau;t)}{|\mathbf{x} + \xi \mathbf{e}_x - \mathbf{X}_i(\tau;t)|^3} \right) \\
&= \partial_t \theta_{\mathrm{sw}}(\mathbf{x},t) + \partial_t \frac{1}{2} \sum_i I_i \int d\tau \frac{\frac{\partial Y_i}{\partial \tau}(z - Z_i(\tau;t)) - \frac{\partial Z_i}{\partial \tau}(y - Y_i(\tau;t))}{(y - Y_i(\tau;t))^2 + (z - Z_i(\tau;t))^2} \\
&\quad \times \left[1 + \frac{x - X_i(\tau;t)}{[(x - X_i(\tau;t))^2 + (y - Y_i(\tau;t))^2 + (z - Z_i(\tau;t))^2]^{1/2}} \right].
\end{aligned} \tag{5.14}$$

Analogous and equivalent expressions can of course be obtained by choosing the integration contour to be along the y- or z-axis.

5.2 Integrating out density fluctuations

It is a standard procedure to "integrate out" the fluctuations in the density ρ in order to obtain an effective theory for the low energy degrees of freedom. For the $U(N)$ Gross-Pitaevskii theory this procedure is described in detail in [8] to which we will refer often in the following. Here we summarize only the main steps and show how to rewrite the calculations using our formalism. The idea is to write $\rho_a = \rho_a^{(0)} + \delta\rho_a$ and then to expand the Lagrangian (5.1) up to second order in $\delta\rho_a$. Then the integral over the $\delta\rho_a$ is purely Gaussian and can be performed analytically. What remains is an effective theory containing only the phase variables θ_a or, alternatively, $\theta_{\mathrm{sw},a}$ and $\boldsymbol{\theta}_{\mathrm{vo},a}$. The expansion of (5.1) up to second order in $\delta\rho_a$ yields

$$\begin{aligned}\mathcal{L}_{\mathrm{SO}} = &-\left(g\rho^{(0)} + \tilde{\partial}_t\theta_a + \frac{1}{2m}(\widetilde{\nabla}\theta_a)^2\right)\delta\rho_a - \frac{1}{2}\delta\rho_a\left(g - \frac{\delta_{ab}}{4m\rho_a^{(0)}}\Delta\right)\delta\rho_b \\ &- \rho_a^{(0)}\left(\tilde{\partial}_t\theta_a + \frac{(\widetilde{\nabla}\theta_a)^2}{2m}\right)\end{aligned} \tag{5.15}$$

where $\rho^{(0)} = \sum_a \rho_a^{(0)}$ and summation over indices is implied here. The term $\rho_a^{(0)}\tilde{\partial}_t\theta_a$ is usually dropped since it is a "total derivative". Moreover, as explained in [8], the term $g\rho^{(0)}$ can be adsorbed by an energy shift of the energy of the zeroth mode. Then, performing the Gaussian integrals, one obtains an effective action

$$S_{\mathrm{eff}} = \frac{1}{2}\int dtd^dxd^dx' J_a(\mathbf{x},t)G_{ab}(\mathbf{x},\mathbf{x}')J_b(\mathbf{x}',t) - \frac{\rho_a^{(0)}}{2m}\int dtd^dx(\widetilde{\nabla}\theta_a(\mathbf{x},t))^2 \tag{5.16}$$

where $J_a(\mathbf{x},t) \equiv \tilde{\partial}_t\theta_a(\mathbf{x},t) + \frac{1}{2m}(\widetilde{\nabla}\theta_a(\mathbf{x},t))^2$ and $G_{ab} = \left(g - \frac{\delta_{ab}}{4m\rho_a^{(0)}}\Delta\right)^{-1}$.
Inserting $\widetilde{\nabla}\theta = \nabla\theta_{\mathrm{sw}} + \nabla\times\boldsymbol{\theta}_{\mathrm{vo}}$ into the second term of (5.16) one can see that for this term the sound wave and vortex part decouple:

$$\int d^3x\left(\nabla\theta_{\mathrm{sw}} + \nabla\times\boldsymbol{\theta}_{\mathrm{vo}}\right)^2 = \int d^3x(\nabla\theta_{\mathrm{sw}})^2 + \int d^3x(\nabla\times\boldsymbol{\theta}_{\mathrm{vo}})^2 \tag{5.17}$$

where the vanishing of the mixed term can be seen by a partial integration. However, for the first term they do not decouple in general. Nevertheless it is a frequently employed assumption that they approximately decouple and one can treat the dynamics of vortices and sound waves separately. In [8] only the sound wave part is considered, i. e. $\nabla\times\boldsymbol{\theta}_{\mathrm{vo}}$ is completely neglected. Then it is possible to simply write $\tilde{\partial}_t\theta = \partial_t\theta_{\mathrm{sw}}$ and $\widetilde{\nabla}\theta = \nabla\theta_{\mathrm{sw}}$. Inserting this into (5.16) yields an action accessible to standard techniques. In the said publication the 2PI effective action is calculated for this action, from which in turn a quantum Boltzmann equation can be derived that can be subjected to a scaling analysis, a procedure which is briefly described in the introduction of this thesis. We state only the

results here. In the large-N limit, one obtains for the scaling exponents in the infrared

$$\beta = 1/2 \tag{5.18}$$
$$\alpha = d/2. \tag{5.19}$$

In the same limit, the spatial scaling exponent κ comes out to be

$$\kappa = d + 1. \tag{5.20}$$

For the case of a single field component, $N = 1$, the resulting exponents are not obvious. In the mentioned publication, the authors propose either $\beta = 1/2$ or $\beta = 1/3$, depending on which term dominates in the scattering integral of the Boltzmann equation. Also for κ they consider two possibilities: $\kappa = d + 3/2$ and $\kappa = d + 2$. It is therefore interesting to compare to numerics.
In the next chapter we want to make the attempt to develop also a description of the vortex part.

5.3 Vortex part

For the treatment of the vortex part we will restrict ourselves to the simplest case of two spatial dimensions and one field component. First one can note that the action (5.16) is still much too complicated for performing calculations in the vortex case. Therefore we make further assumptions in order to arrive at a well-known, much simpler action. First we approximate $G = \left(g - \frac{1}{4m\rho^{(0)}}\Delta\right)^{-1} \approx 1/g$. Furthermore we discard terms that contain spatial and temporal derivatives as well. With this one arrives at

$$S_{\text{eff}} = \int dt\, d^2x \left\{ \frac{1}{2g}\tilde{\partial}_t\theta\tilde{\partial}_t\theta - \frac{\rho^{(0)}}{2m}\widetilde{\nabla}\theta\cdot\widetilde{\nabla}\theta \right\}. \tag{5.21}$$

Introducing the speed of sound $c \equiv \sqrt{g\rho^{(0)}/m}$ and $K \equiv \rho^{(0)}/2mc$ one can rewrite this as

$$S_{\text{eff}} = K\int cdt\, d^2x \left\{ \frac{1}{c^2}\tilde{\partial}_t\theta\tilde{\partial}_t\theta - \widetilde{\nabla}\theta\cdot\widetilde{\nabla}\theta \right\}. \tag{5.22}$$

This action is known under several names in the literature, such as Luttinger liquid action, phason action or continuous XY model (see e. g. [26]). If one assumes the phase field to consist only of a sound wave part, this action is purely Gaussian and therefore trivial. However, if a vortex part is present, it is not.
Here we want to approach this problem in two in some sense complementary ways. First, we will derive from the Luttinger liquid action a "vortex action", containing only the vortex trajectories. The second approach will consist in an attempt to extend the well-known duality between this action and the Sine-Gordon model in 2D (see e. g. [13]) to the 2+1D case. It is easy to see that this cannot work out exactly. However, we will propose arguments why, under certain assumptions, an approximate duality may hold.

5.3.1 Vortex action

As was argued above, if no sound wave part is present we have for $\mathbf{V} = \widetilde{\nabla}\theta$

$$\widetilde{\nabla}\theta = \sum_i q_i \frac{1}{(x - X_i(t))^2 + (y - Y_i(t))^2} \begin{pmatrix} -(y - Y_i(t)) \\ x - X_i(t) \end{pmatrix}. \tag{5.23}$$

For $\tilde{\partial}_t\theta$ we obtain by performing the derivative in (5.9)

$$\tilde{\partial}_t\theta = -\sum_i \dot{\mathbf{X}}_i(t) \cdot \mathbf{V}_i \tag{5.24}$$

where $\mathbf{V}_i$ is the contribution of the ith vortex to $\mathbf{V}$. Inserting this into the Luttinger liquid action yields

$$\begin{aligned} S_{\text{eff}} = &\frac{K}{c^2} \sum_{ij} q_i q_j \int c dt\, d^2x\, \frac{[\dot{Y}_i(x - X_i) - \dot{X}_i(y - Y_i)][\dot{Y}_j(x - X_j) - \dot{X}_j(y - Y_j)]}{[(x - X_i)^2 + (y - Y_i)^2][(x - X_j)^2 + (y - Y_j)^2]} \\ &- K \sum_{ij} q_i q_j \int c dt\, d^2x\, \frac{(x - X_i)(x - X_j) + (y - Y_i)(y - Y_j)}{[(x - X_i)^2 + (y - Y_i)^2][(x - X_j)^2 + (y - Y_j)^2]} \end{aligned} \tag{5.25}$$

where we have suppressed the time dependence of the vortex trajectories for clarity. The idea is now to perform the spatial integrals in order to obtain a Lagrangian containing only the vortex trajectories. This procedure is similar to that employed in [27], with the difference that the authors start from the Gross-Pitaevskii Lagrangian with a constant density, i. e. a Lagrangian linear in $\tilde{\partial}_t\theta$. Since the integrals are UV and IR divergent as well, one has to introduce a short distance and a long distance cutoff a and L, respectively. As is shown in appendix B, assuming $q_i = \pm 1$ one arrives at

$$\begin{aligned} S_{\text{eff}} = &2\pi K \int c dt \Bigg[\sum_i \frac{1}{2} M |\dot{\mathbf{X}}_i|^2 + \sum_{i \neq j} q_i q_j \ln\left(\frac{|\mathbf{X}_j - \mathbf{X}_i|}{L}\right) \\ &- \frac{1}{2c^2} \sum_{i \neq j} q_i q_j \dot{\mathbf{X}}_i \cdot \dot{\mathbf{X}}_j \ln\left(\frac{|\mathbf{X}_j - \mathbf{X}_i|}{e^{1/2} L}\right) \\ &- \frac{1}{2c^2} \sum_{i \neq j} q_i q_j \frac{(\dot{\mathbf{X}}_i \times (\mathbf{X}_j - \mathbf{X}_i))(\dot{\mathbf{X}}_j \times (\mathbf{X}_j - \mathbf{X}_i))}{|\mathbf{X}_j - \mathbf{X}_i|^2} \Bigg] + \text{const} \end{aligned} \tag{5.26}$$

where $M \equiv \ln(L/a)/c^2$, $e = \exp(1)$ and the "$\times$" denotes the two-dimensional (scalar) vector product, i. e. $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$. The first two terms represent the action of a system of charged particles with Coulomb interaction in 2D. In contrast to the Onsager model [28], where the equation of motion is a first-order differential equation, here it is of second order, as in classical mechanics. Additionally, there are two terms which are beyond the pure Coulomb system and will account for new effects. It would be highly interesting to simulate the theory defined by this action, either on a purely classical level or in the truncated Wigner approximation (which to our knowledge has not been performed yet). By comparing to the vortex motion in GP simulations or simulations of other effective theories for vortices such as the Hall-Vinen-Iordanskii equation [29] one could estimate the range of validity of the Luttinger liquid action regarding vortex dynamics. On the other hand, by comparing to Sine-Gordon simulations one could gain insight about the

range of validity of the duality between vortices in the Luttinger liquid model and the Sine-Gordon model outlined in the next section.

5.3.2 Approximate duality to 2+1D Sine-Gordon model

In this section we want to propose arguments why under certain assumptions the well-known duality between vortices in the thermal (2D) Luttinger liquid model and the Sine-Gordon model can be approximatively extended to the real time (2+1D) case. Let us start with the Luttinger liquid action

$$S_{LL} = K \int c dt\, d^2x \left(\frac{1}{c^2} \tilde{\partial}_t \theta \tilde{\partial}_t \theta - \widetilde{\nabla} \theta \cdot \widetilde{\nabla} \theta \right) \equiv K \int c dt\, d^2x\, \tilde{\partial}_\mu \theta \tilde{\partial}^\mu \theta \tag{5.27}$$

where c is the speed of sound and $K = \rho_0/2mc$. In contrast to the previous section we now adopt a point of view where the 2+1D velocity field U_μ (i. e. including the time derivative of θ as 0th component), given by

$$U_\mu = \tilde{\partial}_\mu \theta \tag{5.28}$$

is subjected to a Helmholtz decomposition. Here we keep only the divergence free part, which can be seen as the field created by moving vortices, i. e. vortex tubes in the three dimensional space time. Thus every sound wave contribution is entirely neglected.
Due to the compactness of the phase it must hold for every integral over a closed curve that

$$\oint dx^\mu \tilde{\partial}_\mu \theta = 2\pi n, \;\; n \in \mathbb{Z}. \tag{5.29}$$

This restricts the curl of U to be of the form

$$\epsilon^{\mu\alpha\beta} \partial_\alpha U_\beta(x) = 2\pi \sum_i I_i \int d\tau \delta^{(2+1)}(x^\mu - X_i^\mu(\tau)) \dot{X}_i^\mu \tag{5.30}$$

where the $X_i^\mu(\tau)$ are the parametrizations of the vortex tubes and the I_i are their integer currents. Here we will restrict ourselves to $I_i = \pm 1$. This can be simplified by taking the time component of $X^\mu(\tau)$ to be its curve parameter, whereby we obtain

$$\left(\epsilon^{\mu\alpha\beta} \partial_\alpha U_\beta(t, \mathbf{x})\right) = 2\pi \sum_i I_i \begin{pmatrix} 1 \\ \dot{X}_i/c \\ \dot{Y}_i/c \end{pmatrix} \delta^{(2)}\left(\mathbf{x} - \mathbf{X}_i(t)\right) \tag{5.31}$$

where $\mathbf{X}_i(t)$ now is the trajectory of vortex i (with X_i and Y_i its components) and the dot denotes the time derivative. This parametrization of X^μ by its 0th component is not generally possible but relies on the assumption that every vortex tube crosses every "time layer" only once. This means that we have distinguishable vortex trajectories persisting for the entire time range. Therefore annihilation processes of vortices and antivortices are not included explicitly in this approach. Nevertheless they might be included implicitly: a vortex and an antivortex could approach each other and after having encountered follow the same trajectory, meaning that effectively they have vanished. Though the results presented in [16] strongly suggest that annihilation processes are crucial for scaling in vortex systems, the question whether they are accounted for by our approach must remain

open.
Now we write down a partition function analogous to the equilibrium case, i. e. summing over all vortex numbers and all vortex charges, including a weight ζ. However, the ordinary integrals over the vortex positions in the equilibrium case are now replaced by path integrals over the vortex trajectories:

$$Z = \sum_{n=0}^{\infty} \frac{(-i\zeta)^n}{n!} \sum_{\{I_i=\pm 1\}} \int \prod_{i=0}^{n-1} D\mathbf{X}_i(t) \exp\left(iK \int cdt\, d^2x\, \tilde{\partial}_\mu \theta \tilde{\partial}^\mu \theta \right) \tag{5.32}$$

with

$$\int D\mathbf{X}_i(t) = \mathcal{N} \int \prod_{j=0}^{N-1} d^2 X_i^{(j)}, \tag{5.33}$$

i. e. we slice the time into N time points t_j with distance ϵ_t/c and $\mathbf{X}_i^{(j)}$ is the position of vortex i at time t_j. In the first place one can of course write down any partition function one likes. The above partition function merely takes its inspiration from the equilibrium one and it is not a priori obvious why it should describe the real time system. One has then to argue afterwards that it describes indeed the physics we are considering (at least approximatively). We will postpone this discussion to the end of this section, then also giving an interpretation for ζ.
Since the normalization constant of the path integral $\mathcal{N}$ could be absorbed into ζ and we are a priori free to choose for ζ whatever we like, in principle we are also free to choose for $\mathcal{N}$ whatever we like. It will later turn out to be convenient to choose

$$\mathcal{N} = \mathcal{V}^{-N} \tag{5.34}$$

with $\mathcal{V}$ being the volume of the system.
Now let us perform a Hubbard-Stratonovich transform. To that end we first write the action in discretized form:

$$S_{LL}^{\text{discretized}} = K\epsilon_s^2 \epsilon_t \tilde{\partial}_\mu \theta \circ \tilde{\partial}^\mu \theta = K\epsilon_s^2 \epsilon_t U_\mu \circ U^\mu \tag{5.35}$$

where $\circ$ denotes the "functional scalar product" $\phi \circ \psi \equiv \sum_{t,\mathbf{x}} \phi(t,\mathbf{x})\psi(t,\mathbf{x})$ and ϵ_s and ϵ_t/c are the spatial and temporal discretization, respectively. Consider now the Fresnel formula

$$\int_{-\infty}^{\infty} dy\, \exp(-iay^2 + iJy) = \sqrt{\frac{\pi}{ia}} \exp\left(i\frac{J^2}{4a} \right) \tag{5.36}$$

where $a, J \in \mathbb{R}$, $a \neq 0$. This generalizes trivially to

$$\int_{-\infty}^{\infty} d^n y\, \exp(-ia\vec{y}^2 + i\vec{J}\cdot\vec{y}) = \sqrt{\frac{\pi}{ia}}^n \exp\left(i\frac{\vec{J}^2}{4a} \right) \tag{5.37}$$

for n-dimensional real vectors $\vec{y}$ and $\vec{J}$. When taking this formula to perform the Hubbard-Stratonovich transform, there is a subtlety: on the left hand side we have $-i$ as prefactor,

on the right hand side i (or vice versa by taking $a \to -a$), meaning that in the resulting dual partition function we have e^{-iS} instead of e^{iS}. This implies that the dual action yields backward propagation in time (or negative energies).
We use the Fresnel formula to introduce an auxiliary vector field b^μ:

$$Z = \sum_{n=0}^{\infty} \frac{(-i\zeta)^n}{n!} \int Db^\mu \sum_{\{I_i=\pm 1\}} \int \prod_{i=0}^{n-1} D\mathbf{X}_i(t) \exp\left(-i\frac{1}{4K\epsilon_s^2\epsilon_t} b_\mu \circ b^\mu + i b_\mu \circ U^\mu\right) \quad (5.38)$$

where

$$\int Db^\mu = \sqrt{\frac{i}{4\pi K\epsilon_s^2\epsilon_t}}^{3M} \int \prod_{i=0}^{M-1} \prod_{\alpha=0}^{2} db_i^\alpha \quad (5.39)$$

with M the number of space time lattice points. Now perform a change of variables and introduce a field ϕ^μ with $b^\mu = \epsilon_s^2 \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta/2\pi$ and $\partial_\mu\phi^\mu = 0$. Of course in principle b^μ will also have a curl free part. But if one introduces such a curl free part one will see that it entirely decouples and can be integrated out. This is due to the fact that U^μ is divergence free. Inserting this and casting everything back into the continuum version yields

$$\begin{aligned} Z = &\sum_{n=0}^{\infty} \frac{(-i\zeta)^n}{n!} \int D'\phi^\mu \sum_{\{I_i=\pm 1\}} \int \prod_{i=0}^{n-1} D\mathbf{X}_i(t) \\ &\times \exp\left(-i\frac{1}{16\pi^2 K\epsilon_t^2} \int cdt\, d^2x\, \epsilon_{\mu\alpha\beta}\partial^\alpha\phi^\beta\epsilon^{\mu\gamma\delta}\partial_\gamma\phi_\delta\right) \\ &\times \exp\left(i\frac{1}{2\pi\epsilon_t} \int cdt\, d^2x\, U_\mu \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta\right). \end{aligned} \quad (5.40)$$

The notation $D'\phi$ means that we integrate only over field configurations that fulfill $\partial_\mu\phi^\mu = 0$ (the same could be accomplished by introducing a gauge fixing term $(\partial_\mu\phi^\mu)^2/2\xi$ with $\xi \to 0$ into the Lagrangian). Note that here, in contrast to the standard case of point vortices in 2D, the discretization does *not* drop out (otherwise the units would not even be correct). We will comment on this fact below.
Let us first consider the first exponential. Making use of partial integration and the identity $\epsilon_{\mu\alpha\beta}\epsilon^{\mu\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$, we arrive at

$$\exp\left(-i\frac{1}{16\pi^2 K\epsilon_t^2} \int cdt\, d^2x\, \{-\phi_\mu\Box\phi^\mu + \phi_\mu\partial^\mu(\partial_\nu\phi^\nu)\}\right). \quad (5.41)$$

Taking into account the gauge of ϕ this becomes:

$$\exp\left(-i\frac{1}{16\pi^2 K\epsilon_t^2} \int cdt\, d^2x\, \{-\phi_\mu\Box\phi^\mu\}\right). \quad (5.42)$$

In the second exponential, we move the curl operator by a partial integration from ϕ^μ to U^μ and then insert (5.31):

$$\begin{aligned}
&\exp\left(i\frac{1}{2\pi\epsilon_t}\int cdt\, d^2x\, U_\mu \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta\right)\\
=&\exp\left(i\frac{1}{2\pi\epsilon_t}\int cdt\, d^2x\, \phi_\mu \epsilon^{\mu\alpha\beta}\partial_\alpha U_\beta\right)\\
=&\exp\left(i\frac{1}{\epsilon_t}\int cdt\, d^2x\, \sum_i I_i\left(\phi^0 - \frac{\dot{X}_i}{c}\phi^1 - \frac{\dot{Y}_i}{c}\phi^2\right)\delta^{(2)}\left(\mathbf{x}-\mathbf{X}_i(t)\right)\right)\\
=&\exp\left(i\frac{1}{\epsilon_t}\int cdt\, \sum_i I_i\left(\phi^0(t,\mathbf{X}_i(t)) - \frac{\dot{X}_i}{c}\phi^1(t,\mathbf{X}_i(t)) - \frac{\dot{Y}_i}{c}\phi^2(t,\mathbf{X}_i(t))\right)\right).
\end{aligned} \tag{5.43}$$

Now in order to be able to integrate out the vortex trajectories and to arrive at a sensible QFT one must make two approximations. Though their range of validity is difficult to estimate it seems hard if not impossible to proceed without them. The first one is to say that

$$\phi^0(t,\mathbf{X}_i(t)) - \frac{\dot{X}_i}{c}\phi^1(t,\mathbf{X}_i(t)) - \frac{\dot{Y}_i}{c}\phi^2(t,\mathbf{X}_i(t)) \approx \phi^0(t,\mathbf{X}_i(t)). \tag{5.44}$$

The question is what is about the terms $\dot{X}_i\phi^1/c + \dot{Y}_i\phi^2/c$. Of course one could argue that the speed of the vortices is so low compared to c that one can neglect terms proportional to $\dot{X}_i/c$ and $\dot{Y}_i/c$. However, it is questionable whether one can neglect terms of first order in the vortex velocity, even if the vortices move slowly. With the same argument one could also neglect $\tilde{\partial}_t\theta$ in the first place. A more convincing argument is therefore to assume that $|\phi^0| \gg |\phi^1|, |\phi^2|$. Then it is reasonable to further assume that the dynamics of ϕ^0 is influenced only weakly by the dynamics of ϕ^1 and ϕ^2. In turn the dynamics of ϕ^1 and ϕ^2 would be "driven" by the dynamics of ϕ^0. We will briefly address the assumption $|\phi^0| \gg |\phi^1|, |\phi^2|$ when we will discuss the meaning of ϕ^μ below.
Be that as it may, writing $\phi \equiv \phi^0$ we then arrive at

$$\exp\left(i\frac{1}{\epsilon_t}\int cdt\, \sum_i I_i\phi(t,\mathbf{X}_i(t))\right). \tag{5.45}$$

Casting this back into the discretized form and taking into account the path integrals over the vortex trajectories we have

$$\begin{aligned}
&\sum_{\{I_i=\pm1\}}\int\prod_{i=0}^{n-1}D\mathbf{X}_i(t)\prod_{i=0}^{n-1}\prod_{j=0}^{N-1}\exp\left(iI_i\phi(t_j,\mathbf{X}_i^{(j)})\right)\\
&=\left\{\prod_{j=0}^{N-1}\left(\int\frac{d^2x}{\mathcal{V}}\exp\left(i\phi(t_j,\mathbf{x})\right)\right) + \prod_{j=0}^{N-1}\left(\int\frac{d^2x}{\mathcal{V}}\exp\left(-i\phi(t_j,\mathbf{x})\right)\right)\right\}^n.
\end{aligned} \tag{5.46}$$

Now we have to make the second major approximation. Since we can assume that the system is homogeneous, the expectation value of ϕ is constant in space and we can gauge ϕ such that it is 0. This second approximation consists now in assuming that locally ϕ can fluctuate arbitrarily strongly around its expectation value but averaging over the entire

volume the fluctuation is small. I. e. if we split

$$\int \frac{d^2x}{\mathcal{V}} \exp\left(i\phi(t_j, \mathbf{x})\right) = 1 + \int \frac{d^2x}{\mathcal{V}} (\exp\left(i\phi(t_j, \mathbf{x})\right) - 1) \tag{5.47}$$

we assume that the second term is much smaller than the first one,

$$\int \frac{d^2x}{\mathcal{V}} (\exp\left(i\phi(t_j, \mathbf{x})\right) - 1) \ll 1. \tag{5.48}$$

Since we have not expanded $\exp\left(i\phi(t_j, \mathbf{x})\right)$ by any means, ϕ can still have arbitrarily large fluctuations. The condition is only that this occurs solely in few regions in space. Then we can write the product in (5.46) as

$$\prod_{j=0}^{N-1} \left(1 + \int \frac{d^2x}{\mathcal{V}} (\exp\left(i\phi(t_j, \mathbf{x})\right) - 1)\right) \approx 1 + \sum_{j=0}^{N-1} \int \frac{d^2x}{\mathcal{V}} (\exp\left(i\phi(t_j, \mathbf{x})\right) - 1). \tag{5.49}$$

Discarding irrelevant constants and recasting the result into the continuum version, we get

$$\frac{1}{\epsilon_t \mathcal{V}} \int c dt\, d^2x\, \exp(i\phi(t, \mathbf{x})). \tag{5.50}$$

Doing the analogous calculation for the antivortex contribution and combining the results, we have found the following approximate expression for (5.46):

$$\left(\frac{2}{\epsilon_t \mathcal{V}} \int c dt\, d^2x\, \cos(\phi(t, \mathbf{x}))\right)^n. \tag{5.51}$$

Collapsing now the series over n and combining with the previously found results, we finally obtain for the partition function (omitting irrelevant constants)

$$Z = \int D\phi \exp\left(-i \int c dt\, d^2x \left\{-\frac{1}{16\pi^2 K \epsilon_t^2} \phi \Box \phi + \frac{2\zeta}{\epsilon_t \mathcal{V}} \cos(\phi)\right\}\right). \tag{5.52}$$

As already mentioned, the temporal discretization ϵ_t does not drop out in the end. We would therefore suggest to interpret ϵ_t/c as a characteristic microscopic time scale of the system, e. g. the time in which a vortex with average speed moves by one healing length. Nevertheless the question remains open what is its precise value or whether the fact that it does not drop out even invalidates the entire approach.

Now let us turn to the postponed discussion about the meaning of the partition function, in particular the value of ζ. First one can note that in contrast to the equilibrium case where the grand-canonical ensemble naturally involves a summation over all possible vortex numbers, this summation does not seem to make much sense in the real time case: in a closed system with unitary time evolution the vortex number should be a constant if annihilations are neglected. On the other hand this summation is necessary to obtain the Sine-Gordon field theory in the end. Of course one could also keep only one term in the vortex number series. But apart from the fact that this would yield a rather unhandy QFT for high vortex numbers, this approach has the disadvantage that one usually does not want to specify a precise vortex number (which also depends on the system size) but

rather a vortex density.
Therefore we propose the following line of reasoning: if the prefactor $(-i\zeta)^n/n!$ in the summation (5.32) were removed, all terms would have approximately the same order of magnitude. This can be seen by considering the limit $K \to 0$. Then the normalization constant of the path integral $\mathcal{N} = \mathcal{V}^{-N}$ ensures that without the $(-i\zeta)^n/n!$ prefactor all terms in the series are equal in magnitude. If we now add this prefactor, we can argue that terms with a certain n will give a much larger contribution to Z than others so that effectively the situation in which only one term is kept is restored.
$\zeta^n/n!$ considered as a function of n (seen as a continuous variable) reaches its maximum approximately at $n \approx \zeta$. This can be seen by making use of the Stirling formula

$$\frac{\zeta^n}{n!} \approx \frac{\zeta^n}{\sqrt{2\pi n}(n/e)^n} = \frac{1}{\sqrt{2\pi}} \exp\left(n(\ln\zeta - \ln n + 1) - \frac{1}{2}\ln n\right). \tag{5.53}$$

The position of the maximum can be found by solving for the zero of the derivative:

$$\frac{d}{dn}\left(n(\ln\zeta - \ln n + 1) - \frac{1}{2}\ln n\right) = 0 \tag{5.54}$$

$$\Leftrightarrow \ln\zeta - \ln n - \frac{1}{2n} = 0. \tag{5.55}$$

For $\zeta \gg 1$ this has the approximate solution $n \approx \zeta$. In order to determine also the "width" of $\zeta^n/n!$ we can expand the argument of the exponential in (5.53) to second order around its maximum:

$$\exp\left(n(\ln\zeta - \ln n + 1) - \frac{1}{2}\ln n\right) \approx \exp\left(\zeta - \frac{1}{2}\ln\zeta\right)\exp\left(-\frac{1}{2\zeta}(n-\zeta)^2\right) \tag{5.56}$$

where we have already inserted the approximation $n_{\max} \approx \zeta$ and have furthermore approximated $-1/\zeta + 1/2\zeta^2 \approx -1/\zeta$ (because $\zeta \gg 1$). This shows that the width σ is given by $\sqrt{\zeta}$. Compared to $n_{\max} \approx \zeta$ we have

$$\frac{\sigma}{n_{\max}} \approx \frac{1}{\sqrt{\zeta}} \xrightarrow{\zeta\to\infty} 0. \tag{5.57}$$

So additionally $\zeta^n/n!$ is more and more closely peaked around its maximum for $\zeta \to \infty$. Therefore the terms in the series whose vortex number is around ζ yield the dominant contribution to the partition function, i. e. ζ can be interpreted as the total vortex number. In the action in (5.52) the factor $\zeta/\mathcal{V}$ can thus be interpreted naturally as the mean vortex density ρ_{vo}.
Comparing to the well-known Sine-Gordon Lagrangian

$$\mathcal{L}_{\text{SG}} = -\frac{1}{2\eta}\phi\Box\phi + \lambda\cos(\phi) \tag{5.58}$$

we can now read off the meaning of the parameters η and λ:

$$\eta = 8\pi^2 K\epsilon_t^2 = \frac{4\pi^2\rho_0\epsilon_t^2}{mc} \tag{5.59}$$

$$\lambda = \frac{2\rho_{\text{vo}}}{\epsilon_t}. \tag{5.60}$$

Now let us also discuss the meaning of the field ϕ. To that end consider again the Fresnel formula (5.37). Deriving on both sides by J_i yields

$$\int_{-\infty}^{\infty} d^n y\, y_i \exp(-ia\vec{y}^2 + i\vec{J}\cdot\vec{y}) = \sqrt{\frac{\pi}{ia}}^n \frac{1}{2a} J_i \exp\left(i\frac{\vec{J}^2}{4a}\right), \tag{5.61}$$

deriving by J_i and J_j with $i \neq j$ gives

$$\int_{-\infty}^{\infty} d^n y\, y_i y_j \exp(-ia\vec{y}^2 + i\vec{J}\cdot\vec{y}) = \sqrt{\frac{\pi}{ia}}^n \frac{1}{(2a)^2} J_i J_j \exp\left(i\frac{\vec{J}^2}{4a}\right), \tag{5.62}$$

and so on for higher orders. In our case where $J = U$, $y = b$ and $a = 1/(4K\epsilon_s^2\epsilon_t)$ this implies that

$$\langle b^\mu(x)\rangle = 2K\epsilon_s^2\epsilon_t \langle U^\mu(x)\rangle \tag{5.63}$$
$$\langle b^\mu(x) b^\nu(y)\rangle = (2K\epsilon_s^2\epsilon_t)^2 \langle U^\mu(x) U^\nu(y)\rangle \tag{5.64}$$

and similarly for higher order correlators. Inserting $b^\mu = \epsilon_s^2 \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta/2\pi$ we can write

$$\langle \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta(x)\rangle = 4\pi K\epsilon_t \langle U^\mu(x)\rangle \tag{5.65}$$
$$\langle \epsilon^{\mu\alpha\beta}\partial_\alpha\phi_\beta(x)\epsilon^{\nu\gamma\delta}\partial_\gamma\phi_\delta(y)\rangle = (4\pi K\epsilon_t)^2 \langle U^\mu(x) U^\nu(y)\rangle \tag{5.66}$$
$$\dots$$

This shows that, on the level of expectation values, ϕ^μ can be seen as the vector potential of the velocity field U^μ. Let us explore this in a bit more detail. To ease the notation, we will leave out the averaging brackets $\langle\rangle$ from now on. One should keep in mind, however, that this correspondence between ϕ and the vector potential of the velocity field holds only after averaging over all configurations. Inserting the relation (5.65) into (5.31) we obtain

$$\left(\epsilon^{\mu\alpha\beta}\partial_\alpha(\epsilon_{\beta\kappa\lambda}\partial^\kappa\phi^\lambda)\right) = 8\pi^2 K\epsilon_t \sum_i I_i \begin{pmatrix} 1 \\ \dot{X}_i/c \\ \dot{Y}_i/c \end{pmatrix} \delta^{(2)}\left(\mathbf{x} - \mathbf{X}_i(t)\right). \tag{5.67}$$

Making again use of $\epsilon_{\mu\alpha\beta}\epsilon^{\mu\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$, the left side becomes $-\Box\phi^\mu + \partial^\mu(\partial_\nu\phi^\nu)$. Remembering that we had gauged $\partial_\mu\phi^\mu = 0$, we finally obtain

$$\Box\left(\phi^\mu\right) = -8\pi^2 K\epsilon_t \sum_i I_i \begin{pmatrix} 1 \\ \dot{X}_i/c \\ \dot{Y}_i/c \end{pmatrix} \delta^{(2)}\left(\mathbf{x} - \mathbf{X}_i(t)\right). \tag{5.68}$$

This has the form of the inhomogeneous wave equation, a well-known equation in classical electrodynamics. There one has

$$\Box A^\mu = j^\mu. \tag{5.69}$$

Its general solution is a sum of solutions to the free wave equation and the advanced and retarded potentials. In two spatial dimensions, the retarded potential is given by [30]

$$A^\mu(t,\mathbf{x}) = \frac{1}{2\pi}\int dt' \int d^2x' \frac{\Theta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)}{\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2/c^2}} j^\mu(t',\mathbf{x}'). \tag{5.70}$$

The adiabatic limit, in which retardation effects are neglected, can be obtained by approximating $j^\mu(t',\mathbf{x}') \approx j^\mu(t,\mathbf{x}')$ in the integral:

$$\begin{aligned} A^\mu(t,\mathbf{x}) &\approx \frac{1}{2\pi}\int dt' \int d^2x' \frac{\Theta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)}{\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2/c^2}} j^\mu(t,\mathbf{x}') \\ &= \frac{1}{2\pi}\int_{-t_{\max}}^{-|\mathbf{x}-\mathbf{x}'|/c} dt' \int d^2x' \frac{1}{\sqrt{(t')^2-|\mathbf{x}-\mathbf{x}'|^2/c^2}} j^\mu(t,\mathbf{x}') \\ &= -\frac{1}{2\pi}\int d^2x' \log\left(\frac{|\mathbf{x}-\mathbf{x}'|}{c(t_{\max}+\sqrt{t_{\max}^2-|\mathbf{x}-\mathbf{x}'|^2/c^2})}\right) j^\mu(t,\mathbf{x}'), \end{aligned} \tag{5.71}$$

where we have regularized the integral by introducing a cutoff $t_{\max}$. If $t_{\max} \gg |\mathbf{x}-\mathbf{x}'|/c$, the expression in the denominator becomes approximatively independent of $|\mathbf{x}-\mathbf{x}'|$ and plays the role of a mere integration constant a. Thus we obtain

$$A^\mu(t,\mathbf{x}) \approx -\frac{1}{2\pi}\int d^2x' \log\left(\frac{|\mathbf{x}-\mathbf{x}'|}{a}\right) j^\mu(t,\mathbf{x}'). \tag{5.72}$$

Now we can apply these results to (5.68). We assume that all vortices move much slower than the speed of sound and that we are thus in the adiabatic limit. This yields

$$(\phi^\mu(t,\mathbf{x})) = 4\pi K\epsilon_t \sum_i I_i \log\left(\frac{|\mathbf{x}-\mathbf{X}_i(t)|}{a}\right)\begin{pmatrix}1\\ \dot{X}_i/c\\ \dot{Y}_i/c\end{pmatrix}. \tag{5.73}$$

One can now see that if $\dot{X}_i, \dot{Y}_i \ll c$, we have indeed $|\phi^0| \gg |\phi^1|, |\phi^2|$, as assumed above. $\phi \equiv \phi^0$ is given by

$$\phi(t,\mathbf{x}) = 4\pi K\epsilon_t \sum_i I_i \log\left(\frac{|\mathbf{x}-\mathbf{X}_i(t)|}{a}\right), \tag{5.74}$$

i. e. the electrostatic potential created by the vortex charges. Up to the prefactor this agrees exactly with θ_{vo}, which we introduced previously as the vortex part of the phase and which we are going to calculate numerically in chapter 6.
Finally one should remark that, though our derivation was performed in the real-time formalism, it still does not explicitly take into account the non-equilibrium formalism, which can be seen from our sloppy treatment of boundary terms. In fact a real non-equilibrium derivation would have to treat the boundary conditions much more carefully. We will leave such a more rigorous derivation to further research.

5.4 Relation between phase correlators and fundamental field correlators

In this section we will discuss the relation between correlators of the fundamental Bose fields and the two types of phase correlators regarding scaling. For the case of sound wave phase correlators this subject has already been treated in depth in [8] so that we will only give the main steps for this case while considering the case of vortex phase correlators in more detail. For the sake of simplicity we will restrict ourselves to the one-component case (a generalization to the more-component case is straightforward). It is convenient to work in position space instead of momentum space, so let us translate the scaling relation from momentum space to real space. Say $f(t;\mathbf{k}) = \langle|\psi(\mathbf{k})|^2\rangle$ fulfills $f(t;\mathbf{k}) = (t/t_0)^\alpha f_S((t/t_0)^\beta\mathbf{k})$, then Fourier transforming both sides yields for $g(t;\mathbf{r}) \equiv \mathcal{FT}\{f\}(t;\mathbf{r})$

$$\begin{aligned} g(t;\mathbf{r}) &= (t/t_0)^\alpha \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{r}\cdot\mathbf{k}} f_S((t/t_0)^\beta\mathbf{k}) \\ &= (t/t_0)^{\alpha-d\beta} \int \frac{d^dk}{(2\pi)^d} e^{i((t/t_0)^{-\beta}\mathbf{r})\cdot\mathbf{k}} f_S(\mathbf{k}) \\ &\equiv (t/t_0)^{\alpha-d\beta} g_S((t/t_0)^{-\beta}\mathbf{r}) \end{aligned} \tag{5.75}$$

with $g_S(\mathbf{r})$ the Fourier transform of $f_S(\mathbf{k})$.
So let us now examine the consequences for the scaling behavior of the phase correlator, given that $g(t;\mathbf{r})$ fulfills (5.75). The crucial assumption is that fluctuations in the density are suppressed and we can approximate

$$g(t;\mathbf{r}) \approx \rho^{(0)} \left\langle e^{i\theta(\mathbf{x}+\mathbf{r},t)} e^{-i\theta(\mathbf{x},t)} \right\rangle \equiv \rho^{(0)} C(t;\mathbf{r}). \tag{5.76}$$

In order to express $C(t;\mathbf{r})$ in terms of θ_{sw} and $\boldsymbol{\theta}_{\mathrm{vo}}$ we employ the previously used argumentation: choose a point A sufficiently far away such that it is not affected by the dynamics and $\theta(A)$ is constant. Then the line integral of $\widetilde{\nabla}\theta$ from A to $\mathbf{x}$ will differ from $\theta(\mathbf{x})$ only by a multiple of 2π and an irrelevant constant. Therefore we may write

$$\begin{aligned} C(t;\mathbf{r}) &= \left\langle \exp\left\{ i \int_A^{\mathbf{x}+\mathbf{r}} d\boldsymbol{\ell}\cdot\widetilde{\nabla}\theta - i\int_A^{\mathbf{x}} d\boldsymbol{\ell}\cdot\widetilde{\nabla}\theta \right\} \right\rangle \\ &= \left\langle \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{r}} d\boldsymbol{\ell}\cdot\widetilde{\nabla}\theta \right\} \right\rangle \\ &= \left\langle \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{r}} d\boldsymbol{\ell}\cdot(\nabla\theta_{\mathrm{sw}} + \nabla\times\boldsymbol{\theta}_{\mathrm{vo}}) \right\} \right\rangle \\ &= \left\langle \exp\left\{ i(\theta_{\mathrm{sw}}(\mathbf{x}+\mathbf{r},t) - \theta_{\mathrm{sw}}(\mathbf{x},t)) \right\} \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{r}} d\boldsymbol{\ell}\cdot\nabla\times\boldsymbol{\theta}_{\mathrm{vo}} \right\} \right\rangle . \end{aligned} \tag{5.77}$$

According to the frequently employed assumption that vortex and sound wave dynamics approximately decouple we can factorize this correlator into the sound wave and vortex

part, i. e.

$$C(t;\mathbf{r}) \approx \langle \exp\{i(\theta_{\rm sw}(\mathbf{x}+\mathbf{r},t)-\theta_{\rm sw}(\mathbf{x},t))\}\rangle \left\langle \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{r}} d\boldsymbol{\ell}\cdot\nabla\times\boldsymbol{\theta}_{\rm vo}\right\}\right\rangle . \tag{5.78}$$

In order to have proper scaling of the fundamental Bose field correlator one has further to assume that either the sound wave and the vortex part show the same scaling behavior or that one dominates over the other. In any case it is reasonable to treat them separately. As mentioned above, the sound wave part has been discussed in detail in [8]. The result is that, assuming the absence of accidental cancellations, the scaling relation (5.75) enforces that

$$G^{(n)}_{\rm sw}(t;\{\mathbf{x}_i\})|_{\mathbf{x}_i\in\{0,\mathbf{r}\}} = G^{(n)}_{{\rm sw},S}(\{\mathbf{x}_i\})|_{\mathbf{x}_i\in\{0,(t/t_0)^{-\beta}\mathbf{r}\}} \tag{5.79}$$

where $G^{(n)}_{\rm sw}(t;\{\mathbf{x}_i\})$ is the nth order connected propagator of the sound wave part of the phase, i. e.

$$G^{(n)}_{\rm sw}(t;\{\mathbf{x}_i\}) = \langle \theta_{\rm sw}(\mathbf{x}_1,t)\dots\theta_{\rm sw}(\mathbf{x}_n,t)\rangle_c . \tag{5.80}$$

This means, in particular, that $\langle\theta_{\rm sw}(\mathbf{k})\theta_{\rm sw}(-\mathbf{k})\rangle$ has the same scaling behavior as the fundamental field occupation number $\langle|\psi(\mathbf{k})|^2\rangle$ if the sound wave part dominates.
Now let us turn to the vortex contribution. Assuming particle number conservation for the fundamental Bose field, we may insert $\alpha = d\beta$ so that we have the scaling relation $C(t;\mathbf{r}) = C_S((t/t_0)^{-\beta}\mathbf{r})$. Inserting the representation of $C(t;\mathbf{r})$ in terms of $\boldsymbol{\theta}_{\rm vo}$ into the scaling relation yields

$$\left\langle \exp\left\{ i \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}'\cdot\nabla\times\boldsymbol{\theta}_{\rm vo}(\mathbf{r}',t)\right\}\right\rangle = \left\langle \exp\left\{ i \int_{\mathbf{0}}^{s^{-1}\mathbf{r}} d\mathbf{r}'\cdot\nabla\times\boldsymbol{\theta}_{\rm vo}(\mathbf{r}',t_0)\right\}\right\rangle \tag{5.81}$$

$$\Leftrightarrow \left\langle \exp\left\{ i \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}'\cdot\nabla\times\boldsymbol{\theta}_{\rm vo}(\mathbf{r}',t)\right\}\right\rangle = \left\langle \exp\left\{ i \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}'\cdot\nabla\times\boldsymbol{\theta}_{\rm vo}(s^{-1}\mathbf{r}',t_0)\right\}\right\rangle \tag{5.82}$$

where we have used the short hand notation $s\equiv(t/t_0)^\beta$ and set $\mathbf{x}$ to 0 without loss of generality. ∇ is always acting on $\mathbf{r}'$. Now we can expand on both sides:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbf{0}}^{\mathbf{r}} dx_1^{i_1}\dots dx_n^{i_n}\epsilon_{i_1j_1k_1}\dots\epsilon_{i_nj_nk_n}\frac{\partial}{\partial x_1^{j_1}}\dots\frac{\partial}{\partial x_n^{j_n}}\left\langle\theta^{k_1}_{\rm vo}(\mathbf{x}_1,t)\dots\theta^{k_n}_{\rm vo}(\mathbf{x}_n,t)\right\rangle\\
=&\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbf{0}}^{\mathbf{r}} dx_1^{i_1}\dots dx_n^{i_n}\epsilon_{i_1j_1k_1}\dots\epsilon_{i_nj_nk_n}\frac{\partial}{\partial x_1^{j_1}}\dots\frac{\partial}{\partial x_n^{j_n}}\left\langle\theta^{k_1}_{\rm vo}(s^{-1}\mathbf{x}_1,t_0)\dots\theta^{k_n}_{\rm vo}(s^{-1}\mathbf{x}_n,t_0)\right\rangle
\end{aligned} \tag{5.83}$$

with summation over repeated indices implied and $\theta^k_{\rm vo}$ the components of $\boldsymbol{\theta}_{\rm vo}$. First one can infer from (5.83) that, given that the vortex phase correlators fulfill

$$\left\langle\theta^{k_1}_{\rm vo}(\mathbf{x}_1,t)\dots\theta^{k_n}_{\rm vo}(\mathbf{x}_n,t)\right\rangle = \left\langle\theta^{k_1}_{\rm vo}(s^{-1}\mathbf{x}_1,t_0)\dots\theta^{k_n}_{\rm vo}(s^{-1}\mathbf{x}_n,t_0)\right\rangle \tag{5.84}$$

for all $\{k_i\}$, $C(t;\mathbf{r})$ fulfills $C(t;\mathbf{r}) = C_S(s^{-1}\mathbf{r})$, too. Assuming again the absence of "accidental cancellations" the reverse conclusion can also be established, i. e. from the fulfillment of $C(t;\mathbf{r}) = C_S(s^{-1}\mathbf{r})$ one concludes (5.84). Since connected correlators can always be written as a sum of unconnected propagators, the same scaling relation is then also true for the connected correlators, i. e. we have

$$G_{\mathrm{vo}}^{(n),\{k_i\}}(t;\{\mathbf{x}_i\}) = G_{\mathrm{vo},S}^{(n),\{k_i\}}(\{(t/t_0)^{-\beta}\mathbf{x}_i\}) \tag{5.85}$$

with

$$G_{\mathrm{vo}}^{(n),\{k_i\}}(t;\{\mathbf{x}_i\}) = \left\langle \theta_{\mathrm{vo}}^{k_1}(\mathbf{x}_1,t)\ldots\theta_{\mathrm{vo}}^{k_n}(\mathbf{x}_n,t)\right\rangle_c . \tag{5.86}$$

In particular this means that $\langle \boldsymbol{\theta}_{\mathrm{vo}}(\mathbf{k},t)\cdot\boldsymbol{\theta}_{\mathrm{vo}}(-\mathbf{k},t)\rangle$ shows the same scaling behavior as $\langle|\psi(\mathbf{k})|^2\rangle$, given that the vortex part dominates. The analogous result can be obtained straightforwardly for the two-dimensional case.

6 Numerics of the two-dimensional $U(N)$ system

For the initial condition we used the "box" momentum distribution. I. e. the initial density matrix is given by

$$\hat{\rho}_0 = \prod_{|\mathbf{k}|<Q} \hat{\rho}^{(\mathbf{k})} \prod_{|\mathbf{k}|>Q} |0\rangle_{\mathbf{k}}\langle 0|_{\mathbf{k}} \tag{6.1}$$

with Q the cutoff momentum of the box, $|0\rangle_{\mathbf{k}}$ the vacuum state of mode $\mathbf{k}$ and

$$\hat{\rho}^{(\mathbf{k})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_0 \left|\sqrt{N_0} e^{i\theta_0}\right\rangle_{\mathbf{k}} \left\langle \sqrt{N_0} e^{i\theta_0}\right|_{\mathbf{k}} \tag{6.2}$$

where we denote a coherent state with parameter α in mode $\mathbf{k}$ by $|\alpha\rangle_{\mathbf{k}}$ and N_0 is the number of particles we put into every momentum mode inside the box. I. e. we set the number of particles in the filled modes to a fixed number but average over all phases of the coherent state parameter. Q was chosen to be $\sqrt{2}k_h$ with $k_h = 1/\xi_h$ the healing momentum, as in [15]. For the case of $N > 1$, i. e. more than one field component, we choose the same initial density matrix, given by (6.1), for every single component.
As shown in appendix C, the Wigner function for the filled modes is given in good approximation by

$$\sqrt{\frac{1}{2\pi N_0}} \exp\left(-2\left(\sqrt{N_0} - \sqrt{\rho}\right)^2\right) \tag{6.3}$$

and for the empty modes by

$$2\exp(-2\rho) \tag{6.4}$$

where we write the argument ψ of the Wigner function as $\psi = \sqrt{\rho}e^{i\theta}$.
The Wigner function does not depend on θ. This means that in our Monte-Carlo sampling of the Wigner function every mode gets a phase which is totally random. The modulus receives a Gaussian noise instead. However, for extremely high occupation numbers as are typically chosen in this kind of simulations, this Gaussian noise does not play a very important role. In contrast, the randomness of the phase is essential for obtaining out-of-equilibrium initial dynamics.
The numerical integration of the Gross-Pitaevskii equation (1.4) was performed with the Fourier split-step algorithm.
It turned out to be a central problem of the numerical study of phase correlators that these are objects living in the far IR, i. e. they describe extremely large-scale structures. This requires correspondingly large lattices. E. g. using a 512×512-lattice as employed in [12] turned out not to be sufficient at all since only a "thermal tail" of the phase correlator was visible for this lattice size. Here we generally used a 4096×4096-lattice for the one-component simulation and a 2048×2048 lattice for the three-component case. These huge lattice sizes still were not able to capture the behavior of the phase correlators very well but even larger ones were beyond the scope of the available computational resources. Apart from the long computation time, a limiting factor is also the huge memory requirement of large lattices.

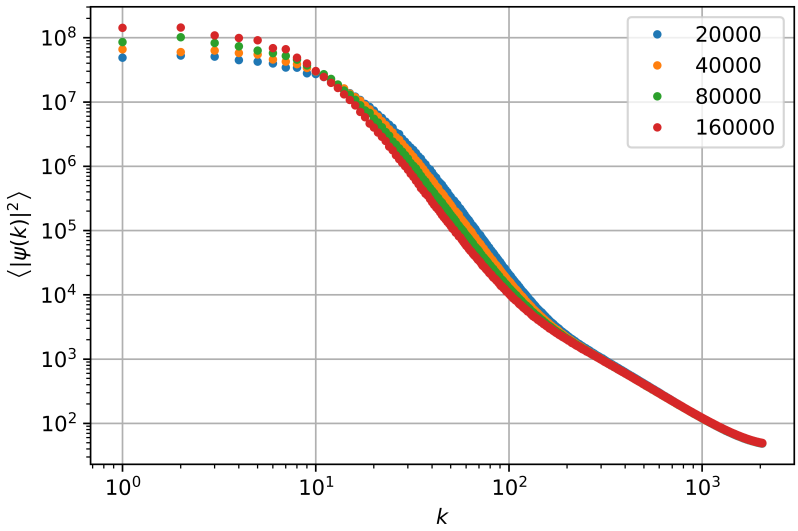

Figure 6.1: Occupation number in the two-dimensional $U(1)$ system.

In the following, all quantities are expressed in numerical units, see appendix A for details.

6.1 One-component system

We chose the following parameters for our simulation, expressed in numerical units:

$$g = 3 \times 10^{-5} \tag{6.5}$$

$$\rho^{(0)} = 2384.2, \tag{6.6}$$

corresponding to 4×10^{10} particles on a 4096×4096 lattice. In general, we averaged over $N = 40$ truncated Wigner runs.
First we consider the usual fundamental Bose field correlator. It is plotted in figure (6.1) (as for the 1D spin system, rescaled plots can be found in the appendix). Fitting the scaling function we obtain

$$\alpha = 0.44 \pm 0.06 \tag{6.7}$$

$$\beta = 0.23 \pm 0.03. \tag{6.8}$$

This agrees well with earlier findings [16, 31]. For β one observes the "anomalous" value $\beta \sim 0.2$ instead of $\beta = 1/2$.
So let us now turn to phase correlators. We consider first the sound wave phase correlator. The result is plotted in figure (6.2). One can immediately see that scaling takes place only in the far infrared. The largest part of the spectrum is non-scaling and characterized by a thermal k^{-2} power law instead. This renders the study of phase correlators numerically tempting since very large lattices are necessary. As in the 1D case there exists no plateau, indicating the lack of a characteristic length scale. However, since here we can see only a small part of the momentum regime where scaling in the phases is present (due to numerical limitations) it is not completely clear whether such a length scaling is entirely absent or simply lies too far in the IR to be observed. A final resolution of this question could only be achieved by employing much stronger computational resources.

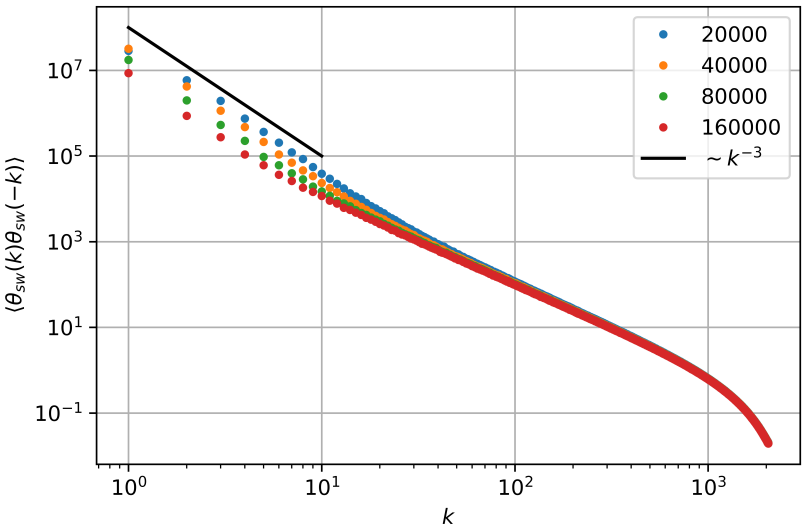

Figure 6.2: Sound wave phase correlator in the two-dimensional $U(1)$ system.

Be that as it may, as in the 1D case we fitted a scaling function containing only one single exponent $\gamma = \alpha - \kappa\beta$. The result is

$$\gamma = -0.89 \pm 0.18. \tag{6.9}$$

Note that here, in contrast to the procedure employed elsewhere in this thesis, we took the latest time as reference time instead of the earliest one due to an apparent outlier in the data of the earliest time.
As can be seen in the plot, $\kappa = 3$ agrees well with the data. However, $\kappa = 7/2$, one of the predictions of the analytical theory, would still be acceptable, too. $\kappa = 4$, another prediction of the analytical theory, seems less likely.
If κ is given and we define α and β by demanding $\alpha = d\beta$, we have $\beta = \gamma/(d - \kappa)$ and $\alpha = d\gamma/(d - \kappa)$. In our case, assuming $\kappa = 3$, this yields

$$\alpha = 1.79 \pm 0.37 \tag{6.10}$$
$$\beta = 0.89 \pm 0.18. \tag{6.11}$$

Assuming instead $\kappa = 7/2$ we obtain

$$\alpha = 1.19 \pm 0.25 \tag{6.12}$$
$$\beta = 0.6 \pm 0.12. \tag{6.13}$$

Interestingly, the β that comes out here seems to be much higher than 0.2 although the fundamental fields scale with $\beta \sim 0.2$. This demonstrates two things: first, this is a further evidence that the anomalous β is caused by the dynamics of vortices; second, one can see that even in a scenario where vortices dominate the sound wave part can have independent dynamics and exhibit scaling on its own.
Unfortunately, these results are in rather poor agreement with the analytical prediction for the sound wave phase correlators outlined in [8]. Depending on which contribution in the quantum Boltzmann equation dominates, the prediction is either $\beta = 1/2$ and $\kappa = 4$

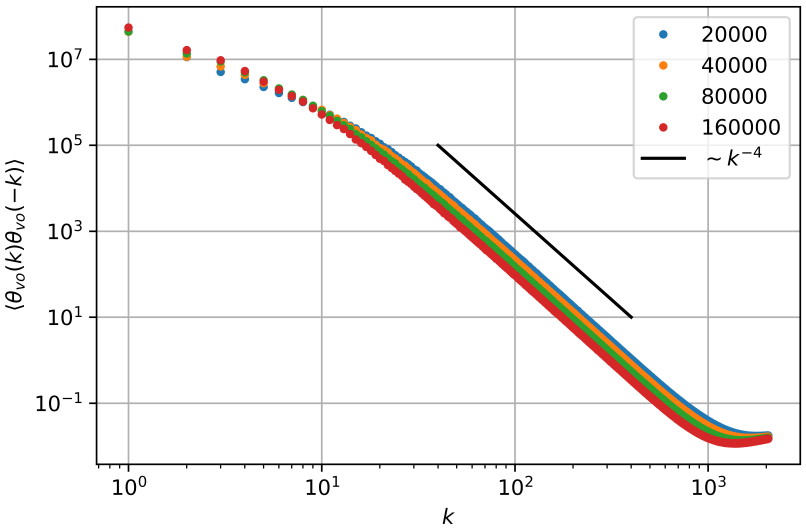

Figure 6.3: Vortex phase correlator in the two-dimensional $U(1)$ system.

or $\beta = 1/3$ and $\kappa = 7/2$. $\kappa = 4$ seems rather unlikely from the data (though it cannot be entirely excluded). Assuming $\kappa = 7/2$ instead yields a β which is close to $1/2$ but not to $1/3$. $\kappa = 3$ gives a rather unusual β close to 1. One should, however, also keep in mind the rather high errors caused by the poorness of the data. Numerical errors could also play a role. This is due to the "delicateness" of the phase observable.
Let us now turn to the vortex phase correlator. It is plotted in figure (6.3). One observes that it follows a very strict k^{-4} power law. However, below a certain momentum threshold, it obeys a thermal k^{-2} power law instead. This is most likely a numerical artifact, caused by a "washing out" of the phase information at a certain distance.
One obtains

$$\gamma = -0.49 \pm 0.06 \tag{6.14}$$

corresponding to

$$\alpha = 0.49 \pm 0.06 \tag{6.15}$$
$$\beta = 0.24 \pm 0.03. \tag{6.16}$$

This small exponents for the vortex part of the phase correlator suggest that the anomalously small exponents which characterize the scaling of the fundamental field correlator are caused by the scaling of the vortex part of the phase.
We want to show that the $\kappa = 4$ power law can be explained by a simple model of randomly distributed vortices. This is similar to the arguments proposed in [12] explaining a k^{-2} power law in the "kinetic energy" of the velocity field, i. e. $\mathbf{V}^2$. Here, however, we want to demonstrate this for $\langle\theta_{\mathrm{vo}}(\mathbf{k})\theta_{\mathrm{vo}}(-\mathbf{k})\rangle$, i. e. correlators of the vector potential of the velocity field. To that end consider the Fourier transform of the velocity field of a

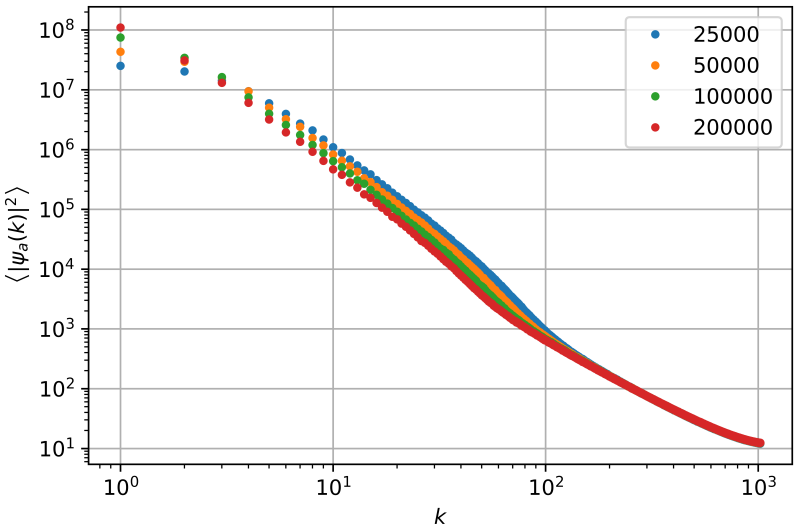

Figure 6.4: Occupation number in the two-dimensional $U(3)$ system.

single vortex located at position $\mathbf{X}$:

$$\begin{aligned}
&\int d^2x \frac{q}{(x-X)^2+(y-Y)^2} \begin{pmatrix} -(y-Y) \\ x-X \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= e^{i\mathbf{k}\cdot\mathbf{X}} \int d^2x \frac{q}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= q e^{i\mathbf{k}\cdot\mathbf{X}} \frac{2\pi i}{k^2} \begin{pmatrix} -k_y \\ k_x \end{pmatrix}
\end{aligned} \tag{6.17}$$

where in the third line we have made use of a result from [12]. Now say we have a system of vortices with positions $\mathbf{X}_i$ and charges $q_i = \pm 1$. Then $\theta_{\text{vo}}(\mathbf{k})\theta_{\text{vo}}(-\mathbf{k})$ is given by

$$\frac{4\pi^2}{k^4} \left[\sum_i 1 + \sum_{i \neq j} q_i q_j e^{i\mathbf{k}\cdot(\mathbf{X}_i - \mathbf{X}_j)} \right]. \tag{6.18}$$

Assuming that the vortices carrying $+1$ and -1 charge are distributed randomly we can average over all possible $\{q_i\}$, which causes the second term in brackets to vanish. The first term yields the total number of vortices, i. e. if ρ_{vo} is the mean vortex density and $\mathcal{V}$ the volume of the system we get

$$\langle \theta_{\text{vo}}(\mathbf{k})\theta_{\text{vo}}(-\mathbf{k}) \rangle = 4\pi^2 \rho_{\text{vo}} \mathcal{V} \frac{1}{k^4}, \tag{6.19}$$

i. e. we obtain $\kappa = 4$.

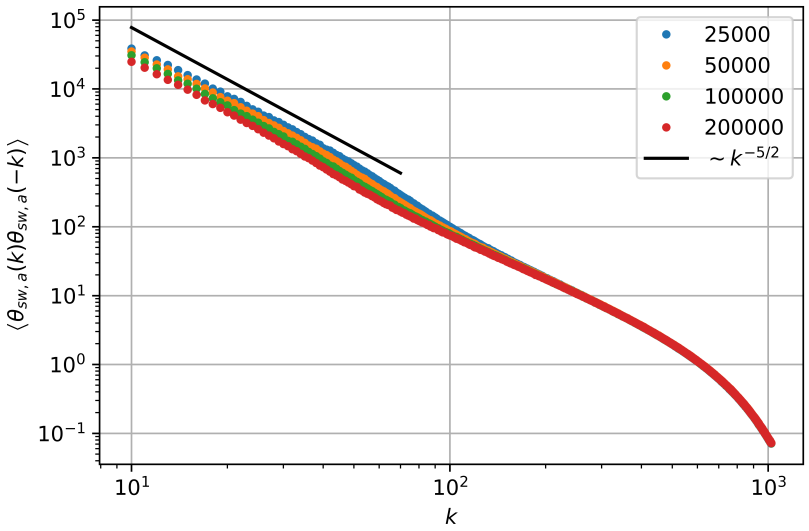

Figure 6.5: Sound wave phase correlator in the two-dimensional $U(3)$ system.

6.2 Three-component system

In the three-component case we chose the same parameters as in the one-component case for every single component:

$$g = 3 \times 10^{-5} \tag{6.20}$$

$$\rho_a^{(0)} = 2384.2, \tag{6.21}$$

corresponding now to 10^{10} particles in each component on a 2048×2048 lattice. We averaged over $N = 25$ truncated Wigner runs. Exploiting the $U(3)$ symmetry of the system we averaged additionally over the three different components.
As in the one-component case we will first consider the fundamental field correlator. To our knowledge, this is also the first time that this system is considered at all (there have been studies of three-component spin systems in 2D [32,33] but due to the additional spin interaction these systems are not really comparable). The plot can be seen in figure (6.4). The fitting procedure yields

$$\alpha = 1.0 \pm 0.06 \tag{6.22}$$

$$\beta = 0.5 \pm 0.03. \tag{6.23}$$

$\beta = 1/2$ is fulfilled to a very high accuracy. This shows that apparently an anomalous fixed point as in the one-component 2D system is not present in the three-component system.
The sound wave phase correlator in the three-component system is plotted in figure (6.5). Similar to the case of the vortex phase correlator in the one-component system, we observed a thermal k^{-2} power law below a certain momentum threshold, probably caused by a "washing out" of the phase by the numerics. This part is cut out in figure (6.5) while the full plot can be found in the appendix. As one can see here, a power law characterized

by $\kappa = 5/2$ fits best to the data. The fitting procedure yields for γ

$$\gamma = -0.27 \pm 0.05. \tag{6.24}$$

With $\kappa = 5/2$ this corresponds to

$$\alpha = 1.07 \pm 0.21 \tag{6.25}$$

$$\beta = 0.53 \pm 0.1. \tag{6.26}$$

This value of β agrees with the analytical prediction. However, κ comes out too small since from the scaling analysis of the Boltzmann equation one would expect $\kappa = 3$ instead of $\kappa = 5/2$.

7 Kinetic theory of the Sine-Gordon model

Motivated by the fact that in a previous section we proposed arguments why the 2+1D Sine-Gordon model might approximately describe the dynamics of vortices, in this section we will make the attempt of developing a kinetic theory of this model in 2+1 dimensions. Apart from the question of vortex dynamics, this is also a subject of interest in itself due to the plethora of applications the Sine-Gordon model has in different branches of physics [34]. We start from the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2\eta}\phi\left(\Box + m^2\right)\phi + \lambda\left(\cos(\phi) + \frac{1}{2}\phi^2 - 1\right) \\ &= -\frac{1}{2\eta}\phi\left(\Box + m^2\right)\phi + \lambda\left(\frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \frac{1}{8!}\phi^8 + \dots\right). \end{aligned} \tag{7.1}$$

In this version of the Lagrangian spontaneous symmetry breaking has already occurred, i. e. $\phi = 0$ in one of the minima of the potential. The η parameter makes sure that the field ϕ is dimensionless so that it can be the argument of the cosine. If there are no additional mass terms in the Lagrangian apart from that in the cosine potential, we have $m^2 = \eta\lambda$. For simplicity we have not inserted the "speed of light" c explicitly here and will set it to 1 in the following. However, one should keep in mind that if the Sine-Gordon model serves as an effective description of some other model, it might have a dependence on several parameters in the original theory. For example in our case it is given by the speed of sound in the Gross-Pitaevskii system, which in turn depends on the mass of the particles, the density and the coupling.

7.1 2PI effective action and self energies

Unfortunately, Sine-Gordon theory does not possess an $O(N)$ symmetry or more than one field at all so that a straightforward $1/N$ expansion is not possible. However, the fact that the $\beta = 1/2$ prediction from $1/N$ resummation was shown to be true also in systems where $N = 1$ [15, 35] seems to suggest that with regard to scaling all that matters is to sum contributions of every order in the coupling constant. Thus our calculation of the 2PI effective action of Sine-Gordon will merely be inspired by $1/N$ resummation in so far as we include only "ring-like" diagrams.
More specifically, we will include only diagrams where every vertex is connected to exactly two neighbors (cf. figure (7.1)) (the case of two vertices is special and must be treated separately because here each vertex can have only one neighbor). Furthermore we will not include "tadpole" contributions, i. e. diagrams where a vertex is connected to itself. In principle the number of connecting propagators between two vertices can be arbitrarily high. However, as we are treating a Lagrangian with a cosine-potential, we can have only even-fold vertices. This requires that the number of connecting propagators be either even or odd between all vertices. This gives two different contributions to the effective action, $\Gamma_e[G]$ and $\Gamma_o[G]$.
Let us first consider the even case. I. e. we have $n > 2$ vertices that can each be connected by $2, 4, 6...$ propagators to its neighbors. From the expansion of the exponential we get $1/n!$. There are $(n-1)!/2$ possibilities of swapping around the vertices, so the prefactor of the nth term is $(i\lambda)^n/2n$. Let us numerate the vertices by i and denote the number of propagators between vertex $i-1$ and i by $2k_i$. Then the vertex i is $2k_i + 2k_{i+1}$-fold.

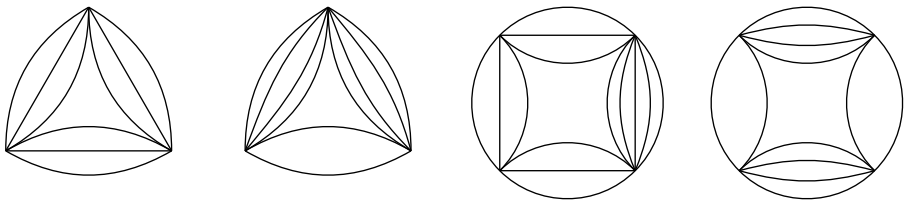

Figure 7.1: Some of the diagrams we include in the computation of the effective action of the Sine-Gordon model. These are still of the "ring" type but the number of propagators between two vertices is arbitrary now, as long as it is always even or always odd.

We have $(2k_i + 2k_{i+1})!/(2k_i)!(2k_{i+1})!$ possibilities to choose which of the outgoing lines we want to connect with vertex $i-1$ and which with vertex $i+1$. Additionally we have $(2k_{i+1})!$ possibilities to permutate the outgoing lines that we have chosen to be connected with vertex $i+1$ (it is important to include this factor only for one of the neighbors of each vertex because otherwise we would count it twice). Finally we have a factor of $(-1)^{k_i+k_{i+1}}/(2k_i + 2k_{i+1})!$ from the expansion of the cosine in the Lagrangian. But in the final product the total sign factor will be $(-1)^{2\Sigma_i k_i} = 1$, so we can discard the factor $(-1)^{k_i+k_{i+1}}$ (in contrast in the odd case the sign factors give a total $(-1)^n$). Putting everything together yields a combinatorial factor of $1/(2k_i)!$ for vertex i. In total we have

$$\begin{aligned}\Gamma_e[G] &= -i\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{2n}\int dx_1\dots dx_n\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\prod_{i=0}^{n}\frac{1}{(2k_i)!}G(x_i,x_{i+1})^{2k_i}\\ &= \frac{i}{2}\mathrm{Tr}\ln_M\left\{\delta(x-y)-i\lambda(\cosh(G(x,y))-1)\right\}. \end{aligned} \tag{7.2}$$

Here $\ln_M$ is the logarithm considered as a matrix function whereas the hyperbolic cosine is a function in the normal sense. In order to get a nice and compact expression, the sum starts at $n=1$ although above we excluded the case $n<3$. One can collect the corrections due to this fact in a $\Gamma_{\text{low-order}}[G]$ that we will neglect here.
Let us now consider the odd case. This case is more involved due to the 2PI requirement: if there is only one pair of vertices connected by only one propagator but all the others are connected by at least three propagators, we still have 2PI, but more than one such pair is not allowed. However, for simplicity we will neglect such diagrams where one single connection is one-fold and assume that the number of connecting propagators is always ≥ 3. Then we get analogously to the even case

$$\begin{aligned}\Gamma_o[G] &= -i\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{2n}\int dx_1\dots dx_n\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}(-1)^n\prod_{i=0}^{n}\frac{1}{(2k_i+1)!}G(x_i,x_{i+1})^{2k_i+1}\\ &= \frac{i}{2}\mathrm{Tr}\ln_M\left\{\delta(x-y)+i\lambda(\sinh(G(x,y))-G(x,y))\right\}. \end{aligned} \tag{7.3}$$

From these expressions for the effective actions one can now calculate the self-energies. We will use the same notation as in the introduction which in turn stems from [7]. The calculations are similar to those for ϕ^4 theory, one only has to replace $G(x,y)^2$ by $\cosh(G(x,y))-1$ and $\sinh(G(x,y))-G(x,y)$, respectively. In this way one gets

$$\overline{\Sigma}_e(x,y) = \lambda\sinh(G(x,y))I_e(x,y) \tag{7.4}$$

$$\overline{\Sigma}_o(x,y) = -\lambda(\cosh(G(x,y)-1)I_o(x,y), \tag{7.5}$$

where the I-functions fulfill:

$$I_e(x,y) = -\lambda(\cosh(G(x,y)) - 1) + i\lambda \int_z I_e(x,z)(\cosh(G(z,y)) - 1) \tag{7.6}$$

$$I_o(x,y) = \lambda(\sinh(G(x,y)) - G(x,y)) - i\lambda \int_z I_o(x,z)(\sinh(G(z,y)) - G(z,y)). \tag{7.7}$$

Now we have to perform the decomposition into ρ- and F-components. Therefor it is useful to use the identities

$$\begin{aligned}
\cosh(G(x,y)) &= \cosh\left(F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0)\right) \\
&= \cosh(F(x,y))\cos(\rho(x,y)/2) \\
&\quad - i\sinh(F(x,y))\sin(\rho(x,y)/2)\,\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0) \qquad (7.8) \\
\sinh(G(x,y)) &= \sinh\left(F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0)\right) \\
&= \sinh(F(x,y))\cos(\rho(x,y)/2) \\
&\quad - i\cosh(F(x,y))\sin(\rho(x,y)/2)\,\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0). \qquad (7.9)
\end{aligned}$$

Using these identities and following the analogous procedure as for the ϕ^4 theory, we obtain the spectral and statistical components of the self-energy in the even case

$$\begin{aligned}
\Sigma_{F,e}(x,y) =& \lambda[\sinh(F(x,y))\cos(\rho(x,y)/2)I_{F,e}(x,y) \\
&- \frac{1}{2}\cosh(F(x,y))\sin(\rho(x,y)/2)I_{\rho,e}(x,y)] \qquad (7.10) \\
\Sigma_{\rho,e}(x,y) =& \lambda[2\cosh(F(x,y))\sin(\rho(x,y)/2)I_{F,e}(x,y) \\
&+ \sinh(F(x,y))\cos(\rho(x,y)/2)I_{\rho,e}(x,y)] \qquad (7.11)
\end{aligned}$$

and in the odd case

$$\begin{aligned}
\Sigma_{F,o}(x,y) =& -\lambda[(\cosh(F(x,y))\cos(\rho(x,y)/2) - 1)I_{F,o}(x,y) \\
&- \frac{1}{2}\sinh(F(x,y))\sin(\rho(x,y)/2)I_{\rho,o}(x,y)] \qquad (7.12) \\
\Sigma_{\rho,o}(x,y) =& -\lambda[2\sinh(F(x,y))\sin(\rho(x,y)/2)I_{F,o}(x,y) \\
&+ (\cosh(F(x,y))\cos(\rho(x,y)/2) - 1)I_{\rho,o}(x,y)]. \qquad (7.13)
\end{aligned}$$

The expressions for the I_ρ and I_F functions are the same as in ϕ^4 theory, one only has different summation functions Π_ρ and Π_F:

$$\Pi_{F,e}(x,y) = -\lambda(\cosh(F(x,y))\cos(\rho(x,y)/2) - 1) \tag{7.14}$$

$$\Pi_{\rho,e}(x,y) = -2\lambda\sinh(F(x,y))\sin(\rho(x,y)/2) \tag{7.15}$$

$$\Pi_{F,o}(x,y) = \lambda(\sinh(F(x,y))\cos(\rho(x,y)/2) - F(x,y)) \tag{7.16}$$

$$\Pi_{\rho,o}(x,y) = \lambda(2\cosh(F(x,y))\sin(\rho(x,y)/2) - \rho(x,y)). \tag{7.17}$$

7.2 $\sin(\rho/2)$ and $\cos(\rho/2)$ in momentum space

In this chapter we want to discuss how to treat the terms $\sin(\rho(x,y)/2)$ and $\cos(\rho(x,y)/2)$ showing up in the self energies obtained above. In particular, we want to find their momen-

tum space representation, which will be necessary for obtaining the Boltzmann integrals. In first-order gradient expansion ρ does not depend on central time. Furthermore the on-shell approximation assumes that we can insert the free spectral function for ρ. In momentum space it is given by

$$\rho(p) = 2\pi\eta \frac{\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2}) - \delta(p^0 + \sqrt{\mathbf{p}^2 + m^2})}{2\sqrt{\mathbf{p}^2 + m^2}}. \tag{7.18}$$

Now we want to determine it in position space (with $t \equiv x^0 - y^0$ and $r \equiv |\mathbf{x} - \mathbf{y}|$):

$$\begin{aligned}
\rho(t, r) &= i \int \frac{d^{2+1}p}{(2\pi)^3} e^{-i(p^0 t - \mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))} 2\pi\eta \frac{\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2}) - \delta(p^0 + \sqrt{\mathbf{p}^2 + m^2})}{2\sqrt{\mathbf{p}^2 + m^2}} \\
&= \eta \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{\sin(\sqrt{\mathbf{p}^2 + m^2}t)}{\sqrt{\mathbf{p}^2 + m^2}} \\
&= \frac{\eta}{2\pi} \int_0^\infty dp J_0(pr) p \frac{\sin(\sqrt{p^2 + m^2}t)}{\sqrt{p^2 + m^2}}.
\end{aligned} \tag{7.19}$$

Surprisingly this last integral permits a simple analytic solution. With the help of formula (8.7.20) of [36] one arrives at

$$\rho(t, r) = \frac{\eta}{2\pi} \mathrm{sgn}(t) \Theta(t^2 - r^2) \frac{\cos(m\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}}. \tag{7.20}$$

Now we have to perform the Fourier transform of $\sin(\rho(x, y)/2)$ and $\cos(\rho(x, y)/2)$. Writing $s \equiv x - y$, $\omega \equiv p^0$ and $p \equiv |\mathbf{p}|$ we have:

$$\begin{aligned}
\frac{1}{i}\mathcal{FT}\left\{\sin(\rho(s)/2)\right\} &= \frac{1}{i} \int d^{2+1}s e^{ip\cdot s} \sin(\rho(s)/2) \\
&= 4\pi \int_0^\infty dt \int_0^\infty dr J_0(pr) r \sin(\omega t) \sin\left(\eta \frac{\Theta(t^2 - r^2)\cos(m\sqrt{t^2 - r^2})}{4\pi\sqrt{t^2 - r^2}}\right)
\end{aligned} \tag{7.21}$$

$$\begin{aligned}
\mathcal{FT}\left\{\cos(\rho(s)/2)\right\} &= \int d^{2+1}s e^{ip\cdot s} \cos(\rho(s)/2) \\
&= 4\pi \int_0^\infty dt \int_0^\infty dr J_0(pr) r \cos(\omega t) \cos\left(\eta \frac{\Theta(t^2 - r^2)\cos(m\sqrt{t^2 - r^2})}{4\pi\sqrt{t^2 - r^2}}\right).
\end{aligned} \tag{7.22}$$

These integrals look hardly analytically solvable. We have therefore solved them numerically. Therefor it is useful to note that not all combinations of the parameters η and m have to be checked since it is only the product $m\eta$ which determines the behavior of the

integral. This can be seen by rescaling by a factor of γ, i. e. $m \to m/\gamma$, $\eta \to \gamma\eta$. Then

$$\begin{aligned} &4\pi \int_0^\infty dt \int_0^\infty dr J_0(pr) r \sin(\omega t) \sin\left(\gamma\eta \frac{\Theta(t^2-r^2)\cos(m/\gamma\sqrt{t^2-r^2})}{4\pi\sqrt{t^2-r^2}}\right) \\ &= \gamma^3 4\pi \int_0^\infty dt \int_0^\infty dr J_0((\gamma p)r) r \sin((\gamma\omega)t) \sin\left(\eta \frac{\Theta(t^2-r^2)\cos(m\sqrt{t^2-r^2})}{4\pi\sqrt{t^2-r^2}}\right) \end{aligned} \tag{7.23}$$

and analogously for $\mathcal{FT}\{\cos(\rho(s)/2)\}$.
The numerical evaluation is performed by discretizing the integration variables r and t in (7.21) and (7.22) with a spacing ϵ and introducing an upper integration limit R. This corresponds to introducing an IR-cutoff $\Lambda_{IR} \sim 1/R$ and an UV-cutoff $\Lambda_{UV} \sim 1/\epsilon$. One has to make sure that they are far away from the characteristic momentum scales m and $1/\eta$, i. e.

$$\Lambda_{IR} \ll m, 1/\eta \ll \Lambda_{UV}. \tag{7.24}$$

The interpretation of the numerical results is highly intricate since we have four parameters (m, η, ϵ and R) and the entire ω-p plane to look at (see appendix D for plots). Additionally the results are expected to be δ-function like. At the first glance it seems that we can well approximate

$$\frac{1}{i}\mathcal{FT}\{\sin(\rho(s)/2)\} \approx \frac{1}{i}\mathcal{FT}\{\rho(s)/2\} = \rho(p)/2 \tag{7.25}$$

$$\mathcal{FT}\{\cos(\rho(s)/2)\} \approx \mathcal{FT}\{1\} = (2\pi)^3\delta^{(2+1)}(p). \tag{7.26}$$

However, having a closer look reveals that especially the case of the approximation (7.25) is a bit subtle. In fact it turns out that instead of $\rho(p)/2$ we rather get

$$\begin{aligned} \frac{\pi\eta}{2\sqrt{\mathbf{p}^2+m^2}}\Big[&\Big(\delta(p^0-\sqrt{\mathbf{p}^2+m^2}) - \delta_\eta(p^0-\sqrt{\mathbf{p}^2+m^2})\Big) \\ &- \Big(\delta(p^0+\sqrt{\mathbf{p}^2+m^2}) - \delta_\eta(p^0+\sqrt{\mathbf{p}^2+m^2})\Big)\Big] \end{aligned} \tag{7.27}$$

where $\delta_\eta(x)$ approaches $\delta(x)$ for $\eta \to \infty$ so that the entire expression approaches 0 in this limit. Therefore one has to assume that the integration regions in the $(p^0, \mathbf{p})$ space which yield the dominant contribution in the Boltzmann integral in the end fulfill $p^0 \pm \sqrt{\mathbf{p}^2+m^2} \ll 1/\eta$.
Although they are still debatable, in the following we will always assume (7.25) and (7.26) because otherwise a scaling behavior of the resulting Boltzmann equation would hardly be possible.

7.3 Non-relativistic limit and ϕ^4 theory limit

In the following we will consider the non-relativistic limit of Sine-Gordon theory. This is necessary because it has a mass $m > 0$ so that it becomes effectively non-relativistic at low momenta. Since the relativistic dispersion $\sqrt{\mathbf{p}^2+m^2}$ a priori does not scale in $\mathbf{p}$, scaling is only possible in either the non-relativistic ($|\mathbf{p}| \ll m$) or the ultra-relativistic

($|\mathbf{p}| \gg m$) limit. One can obtain the non-relativistic limit by expanding the free spectral function (7.18) to lowest order in $|\mathbf{p}|$. This yields

$$\rho(p) \approx 2\pi \frac{\eta}{2m} \left(\delta(p^0 - m - \frac{\mathbf{p}^2}{2m}) - \delta(p^0 + m + \frac{\mathbf{p}^2}{2m}) \right). \tag{7.28}$$

$f(\mathbf{p})$ is then defined by $f(\mathbf{p}) = f(m + \mathbf{p}^2/2m, \mathbf{p})$.
Another quite instructive limit of Sine-Gordon theory is ϕ^4 theory. To be precise, in the expansion of the $-\cos(\phi)$ potential of the Sine-Gordon Lagrangian the ϕ^4 term has a negative prefactor so that strictly speaking no "ϕ^4 theory" limit of Sine-Gordon exists. The resulting theory would be sick because energy would not be bounded from below. Nevertheless we will denote by this name a limit in which only the first few terms of the cosine series are taken into account (e. g. expanding up to order ϕ^6 we would have again a theory whose energy spectrum is bounded from below).
Let us adopt the approximation $\cos(\rho(x,y)/2) \approx 1$ and $\sin(\rho(x,y)/2) \approx \rho(x,y)/2$ that was at least made plausible above. Then it is straightforward to show that additionally expanding $\sinh(F(x,y)) \approx F(x,y)$ and $\cosh(F(x,y)) \approx 1$ in equation (7.10)–(7.17) will recover the well-known expressions for the self-energies in ϕ^4 theory with a negative coupling λ (up to "quantum terms", i. e. terms that contain less fs and are typically neglected). Whether we are in the perturbative regime of the hyperbolic functions (i. e. in the ϕ^4 limit) or in the non-perturbative regime (i. e. beyond ϕ^4 theory) can be seen from the parameter $F_0 \equiv F(x - y = 0)$. If $F_0 \ll 1$ we have the former case, if $F_0 \gg 1$ the latter.
Using $F(p) \approx f(p)\rho(p)$, $f(p^0, \mathbf{p}) \approx -f(-p^0, \mathbf{p})$ (the infinite contribution from the 1/2-terms will have to be renormalized away anyway) and the expanded form of the spectral function (7.28) one obtains in the non-relativistic case

$$F_0 = \frac{\eta n_0}{m} \tag{7.29}$$

with n_0 the average density of particles. In a relativistic setting F_0 has no simple interpretation.
In the following, we will assume that we are in the regime beyond ϕ^4 theory, i. e. $F_0 \gg 1$, because the ϕ^4 theory results are well-known.

7.4 Scattering Integral

In the following we will always assume $f(p) + 1/2 \approx f(p)$ in order to ease the notation and to avoid renormalization issues. We will also mostly suppress the dependence on the central time t unless necessary. Furthermore we will introduce the shorthand notation for

the Fourier transforms of the hyperbolic and trigonometric functions

$$\Xi_s(p) \equiv \mathcal{FT}\{\sinh(F)\} = \int d^{2+1}s e^{ip\cdot s} \sinh(F(s)) \tag{7.30}$$

$$\Xi_c(p) \equiv \mathcal{FT}\{\cosh(F)\} = \int d^{2+1}s e^{ip\cdot s} \cosh(F(s)) \tag{7.31}$$

$$\Omega_s(p) \equiv \frac{1}{i}\mathcal{FT}\{\sin(\rho/2)\} = \frac{1}{i}\int d^{2+1}s e^{ip\cdot s} \sin(\rho(s)/2) \tag{7.32}$$

$$\Omega_c(p) \equiv \mathcal{FT}\{\cos(\rho/2)\} = \int d^{2+1}s e^{ip\cdot s} \cos(\rho(s)/2) \tag{7.33}$$

and

$$\Omega_s^R(p) \equiv \int d^{2+1}s e^{ip\cdot s} \sin(\rho(s)/2)\Theta(s^0). \tag{7.34}$$

The right hand side of the Boltzmann equation is given by

$$\int_0^\infty \frac{dp^0}{2\pi}[\Sigma_\rho(p)F(p) - \Sigma_F(p)\rho(p)]. \tag{7.35}$$

In the following we will only consider the "even" self energies given by equation (7.10), (7.11) and (7.14), (7.15). In the ϕ^4 limit $F_0 \ll 1$ the "odd" contribution vanishes, in the opposite limit $F_0 \gg 1$ its behavior is very similar to that of the "even" one.
The product in real space is a convolution in momentum space. Furthermore $I_F(p) = \Pi_F(p)v_{\text{eff}}(p)$ and $I_\rho(p) = \Pi_\rho(p)v_{\text{eff}}(p)$. In total we have 4 terms in the scattering integral: $\Sigma_\rho(p)F(p)$ gives 2 and $\Sigma_F(p)\rho(p)$ gives 2. We will denote them by $C^{(a)}[f](p)$, $C^{(b)}[f](p)$, $C^{(c)}[f](p)$ and $C^{(d)}[f](p)$. Without further approximations and assumptions they read

$$\begin{aligned} C^{(a)}[f](p) =& -2\lambda^2 \int_0^\infty \frac{dp^0}{2\pi} f(p)\rho(p) \int_{q_1q_2q_3} \Omega_s(p-q_1-q_2-q_3)\Xi_c(q_1) \\ & \times \left(\Xi_c(q_2)\Omega_c(q_3) - (2\pi)^6\delta^{(2+1)}(q_2)\delta^{(2+1)}(q_3)\right) v_{\text{eff}}(q_2+q_3) \end{aligned} \tag{7.36}$$

$$\begin{aligned} C^{(b)}[f](p) =& -2\lambda^2 \int_0^\infty \frac{dp^0}{2\pi} f(p)\rho(p) \int_{q_1q_2q_3} \Omega_c(p-q_1-q_2-q_3)\Xi_s(q_1) \\ & \times \Xi_s(q_2)\Omega_s(q_3)v_{\text{eff}}(q_2+q_3) \end{aligned} \tag{7.37}$$

$$\begin{aligned} C^{(c)}[f](p) =& \lambda^2 \int_0^\infty \frac{dp^0}{2\pi} \rho(p) \int_{q_1q_2q_3} \Omega_c(p-q_1-q_2-q_3)\Xi_s(q_1) \\ & \times \left(\Xi_c(q_2)\Omega_c(q_3) - (2\pi)^6\delta^{(2+1)}(q_2)\delta^{(2+1)}(q_3)\right) v_{\text{eff}}(q_2+q_3) \end{aligned} \tag{7.38}$$

$$\begin{aligned} C^{(d)}[f](p) =& \lambda^2 \int_0^\infty \frac{dp^0}{2\pi} \rho(p) \int_{q_1q_2q_3} \Omega_s(p-q_1-q_2-q_3)\Xi_c(q_1) \\ & \times \Xi_s(q_2)\Omega_s(q_3)v_{\text{eff}}(q_2+q_3) \end{aligned} \tag{7.39}$$

where

$$v_{\text{eff}}(p) = \frac{1}{|1 + \Pi_R(p)|^2} \tag{7.40}$$

$$\Pi_R(p) = -2\lambda \int_q \Omega_s^R(p-q)\Xi_s(q). \tag{7.41}$$

Now we will simplify this by the assumption $\Omega_s(p) \approx \rho(p)/2$, $\Omega_c(p) \approx (2\pi)^3\delta^{(2+1)}(p)$ and $\Omega_s^R(p) \approx G^R(p)/2$. Furthermore we will assume that $F_0 \gg 1$ such that we can neglect 1 compared to $\cosh(F)$, i. e. the second term in the big brackets compared to the first one. This approximation is not necessary for scaling but eases the notation. In fact the effect of subtracting the product of the two delta functions is to cancel the first term in the expansion (7.50). Occasionally it might be necessary to reinstate it to avoid divergences. This gives:

$$C^{(a)}[f](p) = -\lambda^2 \int_0^\infty \frac{dp^0}{2\pi} f(p)\rho(p) \int_{q_1 q_2} \rho(p-q_1-q_2)\Xi_c(q_1)\Xi_c(q_2)v_{\text{eff}}(q_2) \tag{7.42}$$

$$C^{(b)}[f](p) = -\lambda^2 \int_0^\infty \frac{dp^0}{2\pi} f(p)\rho(p) \int_{q_1 q_2} \rho(p-q_1-q_2)\Xi_s(q_1)\Xi_s(q_2)v_{\text{eff}}(p-q_1) \tag{7.43}$$

$$C^{(c)}[f](p) = \lambda^2 \int_0^\infty \frac{dp^0}{2\pi} \rho(p) \int_{q_1} \Xi_s(p-q_1)\Xi_c(q_1)v_{\text{eff}}(q_1) \tag{7.44}$$

$$C^{(d)}[f](p) = \frac{\lambda^2}{4} \int_0^\infty \frac{dp^0}{2\pi} \rho(p) \int_{q_1 q_2 q_3} \rho(p-q_1-q_2-q_3)\Xi_c(q_1)\Xi_s(q_2)\rho(q_3)v_{\text{eff}}(q_2+q_3) \tag{7.45}$$

and

$$\Pi_R(p) = -\lambda \int_q G^R(p-q)\Xi_s(q). \tag{7.46}$$

7.5 Scaling Analysis

7.5.1 Combination of m terms

As mentioned above, we will only consider the non-relativistic limit in the following and we will do so by using the expanded spectral function (7.28). We will first show how one obtains the non-relativistic limit in ϕ^4 theory in this way and then turn to the case of

Sine-Gordon. So let us consider the example term

$$\begin{aligned}
&\int_0^\infty \frac{dp^0}{2\pi} f(p)\rho(p) \int_{q_1 q_2} \rho(p-q_1-q_2) f(q_1)\rho(q_1) f(q_2)\rho(q_2) v_{\text{eff}}(q_1+q_2) \\
&\propto \int_0^\infty dp^0 f(p) \left(\delta(p^0 - m - \frac{\mathbf{p}^2}{2m}) - \delta(p^0 + m + \frac{\mathbf{p}^2}{2m}) \right) \\
&\quad \times \int_{q_1 q_2} \left(\delta(p^0 - q_1^0 - q_2^0 - m - \frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m}) \right. \\
&\qquad \left. - \delta(p^0 - q_1^0 - q_2^0 + m + \frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m}) \right) \\
&\quad \times f(q_1) \left(\delta(q_1^0 - m - \frac{\mathbf{q}_1^2}{2m}) - \delta(q_1^0 + m + \frac{\mathbf{q}_1^2}{2m}) \right) \\
&\quad \times f(q_2) \left(\delta(q_2^0 - m - \frac{\mathbf{q}_2^2}{2m}) - \delta(q_2^0 + m + \frac{\mathbf{q}_2^2}{2m}) \right) v_{\text{eff}}(q_1+q_2). \qquad (7.47)
\end{aligned}$$

Now we can perform all integrals over the 0-components and use $f(-p^0, \mathbf{p}) \approx -f(p^0, \mathbf{p})$. This will in total give 8 contributions, e. g.

$$\begin{aligned}
&f(\mathbf{p}) \int_{\mathbf{q}_1 \mathbf{q}_2} \left[\delta(-2m + \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{q}_1^2}{2m} - \frac{\mathbf{q}_2^2}{2m} - \frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m}) v_{\text{eff}}(2m + \frac{\mathbf{q}_1^2}{2m} + \frac{\mathbf{q}_2^2}{2m}, \mathbf{q}_1+\mathbf{q}_2) \right. \\
&\left. + \delta(\frac{\mathbf{p}^2}{2m} + \frac{\mathbf{q}_1^2}{2m} - \frac{\mathbf{q}_2^2}{2m} - \frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m}) v_{\text{eff}}(-\frac{\mathbf{q}_1^2}{2m} + \frac{\mathbf{q}_2^2}{2m}, \mathbf{q}_1+\mathbf{q}_2) + \dots \right] f(\mathbf{q}_1) f(\mathbf{q}_2). \qquad (7.48)
\end{aligned}$$

There are two distinct kinds of terms: those like the first one in the big brackets that still contain m-terms in the delta function and those like the second term that do not. Since we are in the non-relativistic limit, all typical momenta $p, q_1, q_2 \ll m$, so we can neglect terms of the first kind because the argument of the delta function can only become 0 for atypical momenta for which the f-functions are extremely small. Therefore everything is scaling again.

7.5.2 Requirement for the scaling of $\sinh(F)$ and $\cosh(F)$

Whereas we could find a simple approximation for the $\Omega_s(p)$ and $\Omega_c(p)$ functions that is perfectly scaling, it is not a priori clear why the $\Xi_s(p)$ and $\Xi_c(p)$ functions, i. e. the Fourier transform of $\sinh(F(x,y))$ and $\cosh(F(x,y))$, should show scaling behavior. One way is to assume $F_0 \ll 1$ so that we can expand the hyperbolic functions to lowest order (ϕ^4 theory limit). However, the case $F_0 \gg 1$ is more interesting because it is beyond ϕ^4. In this case scaling behavior can only be achieved by demanding a suitable condition on the scaling exponents. It will turn out that this condition in the non-relativistic case is nothing else than particle conservation $\alpha = d\beta$.
Written as convolutions in momentum space the Ξ-functions read (neglecting again the

1/2-terms)

$$\Xi_s(p) = f(p)\rho(p) + \frac{1}{3!}\int_{q_1 q_2} f(p-q_1-q_2)\rho(p-q_1-q_2)f(q_1)\rho(q_1)f(q_2)\rho(q_2) + \dots \quad (7.49)$$

$$\Xi_c(p) = (2\pi)^3\delta^{(2+1)}(p) + \frac{1}{2!}\int_{q_1} f(p-q_1)\rho(p-q_1)f(q_1)\rho(q_1) + \dots \quad (7.50)$$

Inserting (7.28) for ρ and performing the integrals over the 0-components one can see that in the expansion of $\Xi_s(p)$ only delta functions containing an odd number of ms can be present whereas in the expansion of $\Xi_c(p)$ only those containing an even number of ms are present. Roughly one could say that in position space $\Xi_s(p)$ contains parts oscillating with a frequency of (approximately) $m, 3m, 5m, \dots$ and $\Xi_c(p)$ with $0, 2m, 4m, \dots$. This makes sure that in the scattering integral we can still "combine" different delta functions in the right way to contain no ms in the end so that we have proper scaling.
Let us assume that we have already "combined" all momenta in a way such as to give delta functions containing no m terms. I. e. F^n in momentum space would give expressions of the kind (omitting prefactors)

$$\int_{\mathbf{q}_1\dots\mathbf{q}_{n-1}} \delta(h(\mathbf{p},\mathbf{q}_1,\dots,\mathbf{q}_{n-1}))f(\mathbf{p}-\mathbf{q}_1-\dots-\mathbf{q}_{n-1})f(\mathbf{q}_1)\dots f(\mathbf{q}_{n-1}) \quad (7.51)$$

where $h(\mathbf{k}_1,\dots,\mathbf{k}_n)$ is a function with the property $h(s\mathbf{k}_1,\dots,s\mathbf{k}_n) = s^2 h(\mathbf{k}_1,\dots,\mathbf{k}_n)$. Making the ansatz $f(t;\mathbf{p}) = t^\alpha f_S(t^\beta\mathbf{p})$ (where we have set $t_0 = 1$ for simplicity) one has

$$t^{n\alpha}\int_{\mathbf{q}_1\dots\mathbf{q}_{n-1}} \delta(h(\mathbf{p},\mathbf{q}_1,\dots,\mathbf{q}_{n-1}))f_S(t^\beta(\mathbf{p}-\mathbf{q}_1-\dots-\mathbf{q}_{n-1}))f_S(t^\beta\mathbf{q}_1)\dots f_S(t^\beta\mathbf{q}_{n-1}). \quad (7.52)$$

Rescaling $\mathbf{p}\to t^{-\beta}\mathbf{p}$ one gets a factor of $t^{-d(n-1)\beta+2\beta+n\alpha} = t^{n(\alpha-d\beta)+(d+2)\beta}$ in front where for clarity we have reinstated a general dimension d. We want to make sure that every single term in the expansion of $\sinh(F)$ and $\cosh(F)$ scales in the same way. Therefore $n(\alpha-d\beta)+(d+2)\beta$ must not depend on n. This can only be achieved by demanding $\alpha = d\beta$ which is nothing else than particle conservation.
Note that in usual ϕ^4 theory one has to postulate scaling and a conservation law as well. Here one gets the conservation law for free if one postulates only scaling. One should also note that in the relativistic (massless) case a similar analysis yields $\alpha = (d-1)\beta$ as necessary conservation law. This is neither particle nor energy conservation. It would therefore be an interesting question whether this conservation law could be realized in such a system.

7.5.3 Scaling of Sine-Gordon theory

With these preliminaries we can now perform the scaling analysis of Sine-Gordon theory. The point is that actually the scaling analysis of the ϕ^4 theory remains perfectly valid (for completeness we will perform it for Sine-Gordon nevertheless) – only the question which term dominates is completely different. The reason is that ϕ^4 theory is just an expanded version of Sine-Gordon – but according to the above demand every term in the expansion of $\sinh(F)$ and $\cosh(F)$ scales like the first one.
In ϕ^4 theory we have also 4 terms if we approximate $\Pi_F = F^2 - 1/4\rho^2 \approx F^2$ in the first

place. In symbolic notation, suppressing all integrals, arguments, prefactors and v_{eff}, the scattering integral then reads

$$2\rho F^3 + \rho F^3 - \rho F^3 + \frac{1}{2}\rho^3 F. \tag{7.53}$$

These terms correspond exactly to the four terms (a), (b), (c), (d) for Sine-Gordon. Interestingly the first three terms scale with the usual $\beta = 1/2$. However, the fourth one will have two fs less in the end and therefore scale with two α less. For $\alpha = d\beta$ and $d = 2$ this gives $\beta = 1/6$. Of course in ϕ^4 theory this fourth term will be extremely subdominant. But in Sine-Gordon the situation is entirely different, at least in the non-perturbative regime where $F_0 \gg 1$: here we do not have $\sinh(F) \approx F$ and $\cosh(F) \approx 1$ but $|\sinh(F)| \approx \cosh(F)$. Therefore the term (d) scaling with $\beta = 1/6$ might be comparable or even dominant compared to (a), (b) and (c). If one compares (d) to (a) and (b) in equation (7.42)–(7.45) and assumes that $\Xi_s(p)$ and $\Xi_c(p)$ have the same order of magnitude one can see that the latter still contain "one f more". But on the other hand (d) has "one ρ more" compared to these terms. We will give an estimate for which might outweigh below.
By construction it does not matter which of the terms in the expansions (7.49) and (7.50) we choose for the scaling analysis – they all scale the same. Therefore we will make the easiest possible choice: perform the analysis for the first term in (7.49) and (7.50), respectively. Let us first consider v_{eff} for which we need Π^R:

$$\begin{aligned}
\Pi_R(p) =& -\lambda \int_q G^R(p-q)\Xi_s(q) \\
=& -\lambda \int_q \frac{\eta}{2m}\left(-\frac{1}{p^0 - q^0 - m - \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon} + \frac{1}{p^0 - q^0 + m + \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon}\right) \\
& \times \left[f(q)\frac{2\pi\eta}{2m}\left(\delta(q^0 - m - \frac{\mathbf{q}^2}{2m}) - \delta(q^0 + m + \frac{\mathbf{q}^2}{2m})\right) + \dots\right] \\
=& \frac{-\lambda\eta^2}{(2m)^2}\int_{\mathbf{q}}\left[\frac{1}{p^0 - \frac{\mathbf{q}^2}{2m} + \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon} - \frac{1}{p^0 + \frac{\mathbf{q}^2}{2m} - \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon} + \dots\right] f(\mathbf{q})
\end{aligned} \tag{7.54}$$

Making the ansatz $f(t;\mathbf{p}) = t^\alpha f_S(t^\beta \mathbf{p})$ one has

$$\frac{-\lambda\eta^2}{(2m)^2}\int_{\mathbf{q}}\left[\frac{1}{p^0 - \frac{\mathbf{q}^2}{2m} + \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon} - \frac{1}{p^0 + \frac{\mathbf{q}^2}{2m} - \frac{(\mathbf{p}-\mathbf{q})^2}{2m} + i\epsilon}\right] t^\alpha f_S(t^\beta \mathbf{p}). \tag{7.55}$$

Rescaling $\mathbf{p} \to t^{-\beta}\mathbf{p}$, $p^0 \to t^{-2\beta}p^0$ one gets a factor of t^α in front. Therefore v_{eff} will give $t^{-2\alpha}$. Now let us turn to the rest of the scattering integrals. We will only perform the analysis for the contributions (b) and (d) (for the remaining ones it is very similar). For

(b) we have (omitting the prefactors)

$$\begin{aligned}
&\int_0^\infty dp^0 f(p)\left(\delta(p^0-m-\frac{\mathbf{p}^2}{2m})-\delta(p^0+m+\frac{\mathbf{p}^2}{2m})\right)\\
&\times\int_{q_1q_2}\left(\delta(p^0-q_1^0-q_2^0-m-\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m})\right.\\
&\qquad\left.-\delta(p^0-q_1^0-q_2^0+m+\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m})\right)\\
&\times\left[f(q_1)\left(\delta(q_1^0-m-\frac{\mathbf{q}_1^2}{2m})-\delta(q_1^0+m+\frac{\mathbf{q}_1^2}{2m})\right)+\dots\right]\\
&\times\left[f(q_2)\left(\delta(q_2^0-m-\frac{\mathbf{q}_2^2}{2m})-\delta(q_2^0+m+\frac{\mathbf{q}_2^2}{2m})\right)+\dots\right]v_{\text{eff}}(p-q_1).
\end{aligned}\tag{7.56}$$

One example term of the many terms one gets is

$$f(\mathbf{p})\int_{\mathbf{q}_1\mathbf{q}_2}\delta(\frac{\mathbf{p}^2}{2m}-\frac{\mathbf{q}_1^2}{2m}+\frac{\mathbf{q}_2^2}{2m}-\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m})v_{\text{eff}}(\frac{\mathbf{p}^2}{2m}-\frac{\mathbf{q}_1^2}{2m},\mathbf{p}-\mathbf{q}_1)f(\mathbf{q}_1)f(\mathbf{q}_2)+\dots$$

Inserting $f(t;\mathbf{p})=t^\alpha f_S(t^\beta\mathbf{p})$ and rescaling $\mathbf{p}\to t^{-\beta}\mathbf{p}$ one obtains in total a factor of $t^{3\alpha-2\cdot2\beta+2\beta-2\alpha}=t^{\alpha-2\beta}$, so the scaling condition yields

$$\alpha-1=\alpha-2\beta\tag{7.57}$$

$$\Leftrightarrow\beta=1/2.\tag{7.58}$$

The same is true for (a) and (c). Now let us turn to (d), which is different. We have

$$\begin{aligned}
&\int_0^\infty dp^0\left(\delta(p^0-m-\frac{\mathbf{p}^2}{2m})-\delta(p^0+m+\frac{\mathbf{p}^2}{2m})\right)\\
&\times\int_{q_1q_2q_3}\left(\delta(p^0-q_1^0-q_2^0-q_3^0-m-\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)^2}{2m})\right.\\
&\qquad\left.-\delta(p^0-q_1^0-q_2^0-q_3^0+m+\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)^2}{2m})\right)\\
&\times\left[(2\pi)^3\delta^{(2+1)}(q_1)+\dots\right]\\
&\times\left[f(q_2)\left(\delta(q_2^0-m-\frac{\mathbf{q}_2^2}{2m})-\delta(q_2^0+m+\frac{\mathbf{q}_2^2}{2m})\right)+\dots\right]\\
&\times\left(\delta(q_3^0-m-\frac{\mathbf{q}_3^2}{2m})-\delta(q_3^0+m+\frac{\mathbf{q}_3^2}{2m})\right)v_{\text{eff}}(q_2+q_3).
\end{aligned}\tag{7.59}$$

Again we consider only one example term:

$$\int_{\mathbf{q}_1\mathbf{q}_2}\delta(\frac{\mathbf{p}^2}{2m}+\frac{\mathbf{q}_1^2}{2m}-\frac{\mathbf{q}_2^2}{2m}-\frac{(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m})f(\mathbf{q}_1)v_{\text{eff}}(-\frac{\mathbf{q}_1^2}{2m}+\frac{\mathbf{q}_2^2}{2m},\mathbf{q}_1+\mathbf{q}_2)+\dots\tag{7.60}$$

Now we get a prefactor of $t^{-\alpha-2\beta}$ so that the scaling condition reads

$$\alpha - 1 = -\alpha - 2\beta \tag{7.61}$$

$$\Leftrightarrow 2\alpha + 2\beta = 1. \tag{7.62}$$

Inserting $\alpha = 2\beta$ this yields $\beta = 1/6$.
One should keep in mind that this sloppy kind of derivation of course is only sufficient for the question of scaling – not the question which term dominates.

7.6 Estimating which term dominates

In a heuristic way we want to derive a very rough condition for the dominance of the term (d) scaling with $\beta = 1/6$ (assuming in the first place that we are in the non-perturbative regime $F_0 \gg 1$). This term has "one ρ more and one f less" compared to the terms (a) and (b). First one can note that in the plateau region $f(\mathbf{p}) \sim n_0\xi^2$ with n_0 the number of particles per volume and ξ the spatial coherence length. Therefore $n_0\xi^2$ is the "enhancement factor" a term gets due to "one f more". Next we have to determine the enhancement factor due to one ρ more. In the non-relativistic case the free spectral function comes with an η/m factor in front. However, in contrast to f (which is dimensionless) η/m has the dimension length squared. So in order to estimate whether it is large or small we have to compare to another length scale squared. The most natural length scale in this context to compare with is the spatial "coherence length" $\bar{\xi}$ of the $\Xi_s(p)$ and $\Xi_c(p)$ functions. So the enhancement factor due to one ρ more would be $\sim \eta/(m\bar{\xi}^2)$. In order that the term with one ρ more dominate its enhancement factor has to be much larger than that of the terms with one f more, i. e.

$$\frac{\eta}{m\bar{\xi}^2} \gg n_0\xi^2 \tag{7.63}$$

$$\Leftrightarrow \frac{\eta}{m} \gg n_0\xi^2\bar{\xi}^2 \tag{7.64}$$

$$\Leftrightarrow \left(\frac{\eta}{\lambda}\right)^{1/2} \gg n_0\xi^2\bar{\xi}^2. \tag{7.65}$$

The question remains what is $\bar{\xi}$. It is almost certain that $\bar{\xi} \ll \xi$ since $\sinh(F)$ and $\cosh(F)$ decay much faster in position space than F, but the exact factor is not obvious. Assume for a moment that the spatial and temporal dependence of F can be analyzed independently (i. e. that F factorizes) and consider only the spatial part. One can make different assumptions on how it looks like. E. g. one could say that it is of the form $\sim F_0 \exp(-|\mathbf{s}|/\xi)$. Then in $\sinh(F) \approx \cosh(F) \approx \exp(F)/2$ one can expand $\exp(F_0 \exp(-|\mathbf{s}|/\xi)) \approx \exp(F_0)\exp(-|\mathbf{s}|/(\xi/F_0))$ and we obtain $\bar{\xi} = \xi/F_0 = m\xi/(\eta n_0)$. However, one could also make a different ansatz for the spatial dependence of F such as $\sim \exp(-|\mathbf{s}|^2/2\xi^2)$ whereby one obtains $\bar{\xi} = \xi/\sqrt{F_0} = \xi\sqrt{m/\eta n_0}$.
What is the "right" choice for $\bar{\xi}$ and whether these heuristic arguments are correct at all remains debatable. It might also be possible that phase space factors are much more important for the dominance of a term. Deciding these questions would require a more precise knowledge of the form of $F(s)$ and a much more thorough treatment of the Boltzmann integrals, probably also numerically. We leave this to further research.

8 Numerical treatment of the Sine-Gordon model

Though it would be highly desirable to simulate the Sine-Gordon model itself, it turned out that this is numerically quite tempting. The reason is that the momentum spectrum is extremely "stretched out", i. e. $f(k) \gg 0$ for a very broad range of momenta. This is likely caused by the fact that one has dynamics at very short distances (dynamics within one minimum of the cosine potential) and at long distances (large-scale structure formation) alike. This is problematic because at the same time one has to make sure that the IR is still sufficiently well resolved for extracting the scaling behavior and that the highest modes on the lattice are not occupied (otherwise one has almost immediate thermalization). This requires very large lattice sizes. We would estimate that lattice sizes of at least 8192×8192 or 16384×16384 would be necessary, preferably even more. This was clearly beyond the range of available computational resources.
Surprisingly, however, we found the non-relativistic limit of Sine-Gordon to be much more feasible numerically. At first one might think that this limit is a "Sine-Pitaevskii" model, i. e. a Gross-Pitaevskii Lagrangian with $\cos(|\psi|)$ instead of $|\psi|^4$, such as Gross-Pitaevskii is the non-relativistic limit of ϕ^4 theory. However, it turns out that this is not quite true but instead the limit is given by the Gross-Pitaevskii Lagrangian with a Bessel function replacing $|\psi|^4$. This is demonstrated in the next section. Thereafter we present numerical results for this theory.
The reasoning behind taking the non-relativistic limit of Sine-Gordon to examine its far from equilibrium dynamics is that at small momenta, relevant for non-thermal fixed points, a relativistic theory becomes effectively non-relativistic anyways. Consider e. g. that also in the last chapter we derived the scaling exponents of Sine-Gordon assuming it to be in its non-relativistic limit. A further advantage apart from the numerical feasibility is that in the non-relativistic limit of Sine-Gordon one can determine quite easily whether a potential scaling with an exponent ~ 0.2 might not be caused by topological defects in Sine-Gordon itself (cf. [11]).

8.1 Non-relativistic limit of the Sine-Gordon equation

Our derivation of the non-relativistic limit of the Sine-Gordon equation will be a generalization of the way in which the Gross-Pitaevskii equation is derived as the non-relativistic limit of the equation of motion of ϕ^4 theory in [37]. Obviously, the only difference lies in the different potential term so that most of the time we can closely follow the derivation in this paper.
Let us start from the equation of motion of the Sine-Gordon model

$$\Box\phi + m^2\phi + \eta\lambda(\sin(\phi) - \phi) = 0 \tag{8.1}$$

with $m^2 = \eta\lambda$. We are in the non-relativistic limit, i. e. $p \ll m$ and $E \approx m + \frac{p^2}{2m}$. The idea is now to factor out fast oscillations with frequency m. Therefore we write

$$\phi = \Re\{\psi \exp(-imt)\} \tag{8.2}$$

with $\psi \in \mathbb{C}$. Since we have factored out the fast oscillations, ψ changes slowly. Thus

$$\left|\frac{\partial^2\psi}{\partial t^2}\right| \ll m\left|\frac{\partial\psi}{\partial t}\right|. \tag{8.3}$$

Now we insert (8.2) into the equation of motion and neglect second derivatives in time:

$$\Re\left\{\left[-2im\frac{\partial\psi}{\partial t}-\Delta\psi\right]\exp(-imt)\right\}+\eta\lambda\left(\sin(\Re\{\psi\exp(-imt)\})-\Re\{\psi\exp(-imt)\}\right)=0. \tag{8.4}$$

The potential term requires a closer inspection:

$$\begin{aligned}
&\sin(\Re\{\psi\exp(-imt)\})-\Re\{\psi\exp(-imt)\}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\Re\{\psi\exp(-imt)\}\right)^{2n+1}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{2^{2n+1}}\left(\psi\exp(-imt)+\psi^*\exp(imt)\right)^{2n+1}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{2^{2n+1}}\sum_{k=0}^{2n+1}\binom{2n+1}{k}\psi^{2n+1-k}(\psi^*)^k\exp(-i(2n+1-2k)mt).
\end{aligned} \tag{8.5}$$

Now we discard all terms oscillating with a frequency larger than m, i. e. we keep only those with $k=n$ and $k=n+1$. This corresponds to neglecting particle number changing processes. Then we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{2^{2n+1}}\frac{(2n+1)!}{n!(n+1)!}|\psi|^{2n}2\Re\{\psi\exp(-imt)\}\\
&=\left(\frac{2}{|\psi|}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{|\psi|}{2}\right)^{2n+1}-1\right)\Re\{\psi\exp(-imt)\}\\
&=\left(\frac{2}{|\psi|}J_1(|\psi|)-1\right)\Re\{\psi\exp(-imt)\}
\end{aligned} \tag{8.6}$$

where $J_1(x)$ is a Bessel function of the first kind. Inserting this into (8.4) we find

$$\Re\left\{\left[-2im\frac{\partial\psi}{\partial t}-\Delta\psi+\eta\lambda\left(\frac{2}{|\psi|}J_1(|\psi|)-1\right)\psi\right]\exp(-imt)\right\}=0 \tag{8.7}$$

from which it follows that

$$-2im\frac{\partial\psi}{\partial t}-\Delta\psi+\eta\lambda\left(\frac{2}{|\psi|}J_1(|\psi|)-1\right)\psi=0, \tag{8.8}$$

or dividing out $-2m$

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2m}\Delta\psi+\frac{\eta\lambda}{m}\left(\frac{1}{|\psi|}J_1(|\psi|)-\frac{1}{2}\right)\psi. \tag{8.9}$$

We will call this the "Bessel-Gross-Pitaevskii" (BGP) equation. The corresponding Lagrangian reads

$$\mathcal{L}_{\rm BGP}=\frac{i}{2\eta^2}(\psi^*\partial_t\psi-\psi\partial_t\psi^*)-\frac{1}{2m\eta^2}\nabla\psi^*\cdot\nabla\psi+\frac{2\lambda}{m\eta}(J_0(|\psi|)+\frac{1}{4}|\psi|^2-1). \tag{8.10}$$

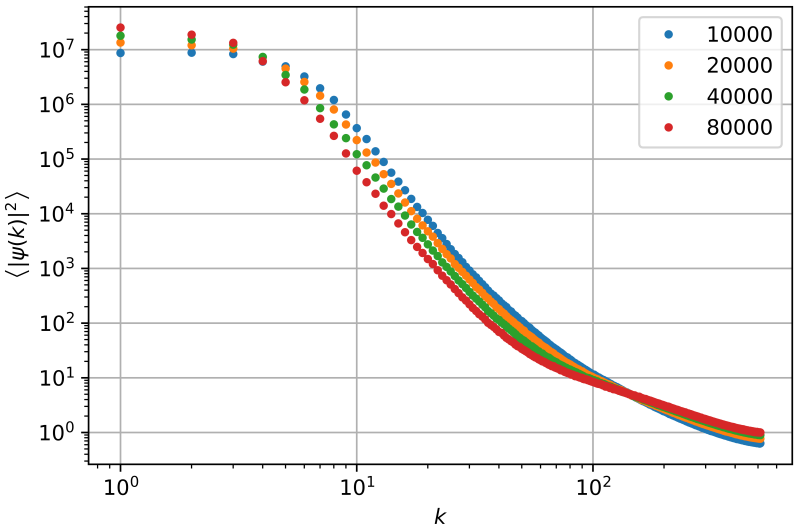

Figure 8.1: The occupation number in the BGP system.

The initial condition in the truncated Wigner simulations is then chosen as

$$\psi(\mathbf{r}) = \frac{\eta}{\sqrt{\mathcal{V}}} \sum_{\mathbf{k}} \psi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{8.11}$$

with the $\psi_{\mathbf{k}}$ chosen as in the normal truncated Wigner simulations of Gross-Pitaevskii. I. e. the only difference to the standard case is the factor η, which ensures that ψ is dimensionless.

8.2 Numerical results

We have simulated the system described by the Lagrangian (8.10) employing truncated Wigner simulations. We chose a 1024×1024 lattice and averaged over $N = 50$ realizations. As for the Gross-Pitaevskii system, we used a "box" initial condition, given by (6.1). We were indeed able to observe a scaling with an anomalously small exponent β, by the following choice of parameters (expressed in numerical units):

$$m = \sqrt{\eta\lambda} = 1 \tag{8.12}$$
$$\eta = 1 \tag{8.13}$$
$$Q = 0.05 \tag{8.14}$$
$$\rho_0 = 10^3. \tag{8.15}$$

If the number of particles were equal at all lattice sites, $\rho_0 = 10^3$ would correspond to a field value of $|\psi| = 10^{1.5} \approx 31.6$. I. e. the average field value would be close to the sixth minimum of the Bessel function. This definitely makes sure that we are not simulating some Gross-Pitaevskii limit of the Bessel-Gross-Pitaevskii model (which could be the case if $|\psi|$ were so small that it never leaves the 0th minimum of the potential).
The resulting occupation number distribution is shown in figure (8.1). For the scaling

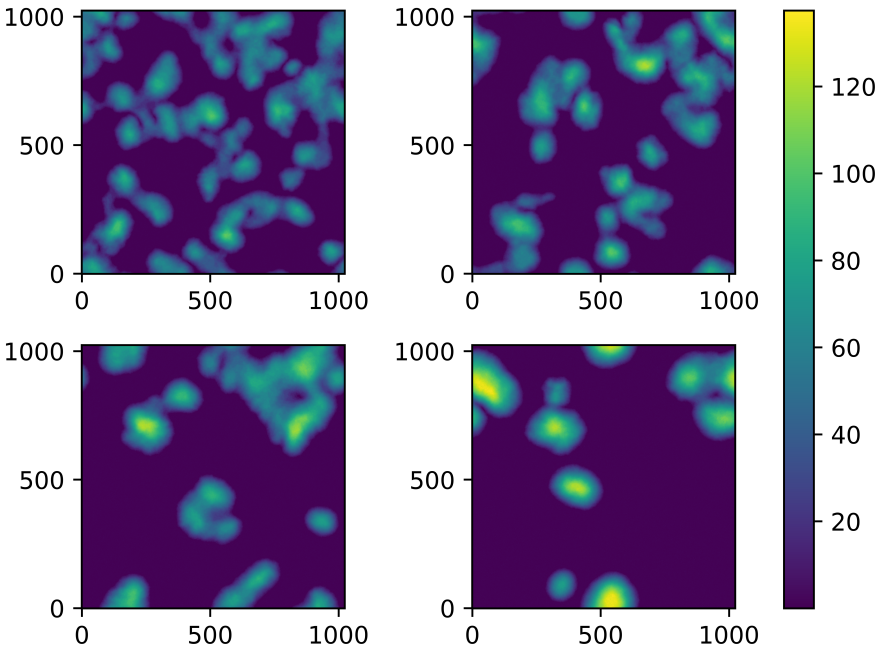

Figure 8.2: $|\psi|$ in real space for one example realization and for the same four times as in the occupation number plot (8.1), with time increasing from left to right and from top to bottom.

exponents we obtain

$$\alpha = 0.53 \pm 0.07 \tag{8.16}$$

$$\beta = 0.24 \pm 0.02. \tag{8.17}$$

This is close to the $\beta = 1/6$ that was shown to be allowed by the Boltzmann equation of the Sine-Gordon model though there is a slight deviation. This deviation might be attributed to the fact that the scattering integral contains also terms scaling with $\beta = 1/2$ that possibly are not entirely suppressed (additionally it seems to be a "rule of thumb" that the numerically extracted exponents are slightly larger than those from the Boltzmann equation, cf. e. g. the results in [8, 15, 38]). A crucial question is of course whether this small scaling exponent might not be caused by topological defects in the Bessel-Gross-Pitaevskii model, too. First one can note that such a topological defect is rather unlikely to be stable if ρ_0 is large since the potential $V(|\psi|) = -J_0(|\psi|)$ would change its sign many times between the core of the defect and the bulk. Additionally, a look at some of the field configurations (figure (8.2)) is able to dispel such concerns even further. In fact, no such things as vortices are visible.
Instead, one observes small patches in space where $|\psi|$ grows large while it sits in the 0th minimum of the Bessel function elsewhere. This can be attributed to the fact that on the one hand the potential $-J_0(|\psi|)$ has an absolute minimum at $|\psi| = 0$ but on the other hand approaches 0 for $|\psi| \to \infty$ ($J_0(x)$ has the asymptotic form $\sqrt{2/\pi x}\cos(x - \pi/4)$ for large x). It is suggestive to identify such small patches of large $|\psi|$ with the phenomenon of clustering of equal charge vortices that was observed in [16] and was made responsible for the anomalously small exponent β there. However, since we were not able to demonstrate the duality between vortices and the Sine-Gordon model rigorously, such an identification must remain speculative.

9 Conclusion and outlook

In this thesis, we have discussed several different ways of defining a correlation function of phases $\langle\theta(x)\theta(y)\rangle$: the obvious definition, in which the phases are kept as they are and jumps are accepted; different modification definitions, in which instead of $\langle\theta(x)\theta(y)\rangle$ one considers $\langle f(\theta(x),\theta(y))\rangle$ with f chosen such that jumps are avoided but the resulting correlator somehow resembles the original one; and finally the continuation definition in which the phase is defined as either the scalar or vector potential of the velocity field $\widetilde{\nabla}\theta$. We subsequently applied these different definitions to a one-dimensional spin system and found that in terms of temporal scaling, i. e. as far as the dynamical exponents α and β are concerned, these definitions all yield the same result, $\alpha, \beta \sim 0.25$, in agreement with the values obtained for the fundamental field. We found furthermore that the obvious definition yields a fractional exponent κ that stems from the jumps in the phase, as can be shown by a simple model of random jumps. In contrast, our proposal for a modification definition and the continuation definition both yielded $\kappa = 2$, in agreement with the fundamental field result. The continuation definition was the only one not to yield a plateau in the IR, indicating coherence on a very large (potentially infinite) length scale. We then concluded that the continuation definition, which introduces two phase fields, the sound-wave and the vortex phase field, is the most natural and suitable definition. Additionally, it is the only one accessible to analytical treatment. In the following we showed how to rewrite earlier findings in terms of these two phase fields, reviewing results for the sound-wave part of the phase. We then tried to develop a description also for the vortex part, by making use of two approaches. We first derived an action containing only the vortex trajectories. The second approach was an attempt to generalize the duality between vortices and the Sine-Gordon model from 2D to 2+1D. We found this duality to approximately hold under several assumptions. Unfortunately, we were not able to determine their range of validity (or whether there are scenarios in which they hold at all).

We then examined the two-dimensional Gross-Pitaevskii system numerically, considering the one-component and three-component case. In the one component case, we found the anomalously small $\beta = 0.23 \pm 0.03$ for the fundamental field correlator, in agreement with earlier findings. In contrast, in the three-component case we obtained $\beta = 0.5 \pm 0.03$. When considering the phase correlator, we encountered some numerical problems related to the "delicateness" of the phase observable. On the one hand, scaling appeared only in the far infrared (at least for the scalar phase correlator in the one-component case), requiring very large lattice sizes. On the other hand, the phase seemed often to be "washed out" by the numerics. For the scalar phase correlator in the one-component system we got $\gamma = -0.89 \pm 0.18$ and κ between 3 and 3.5. Assuming $\kappa = 3.5$, this yields $\beta = 0.6 \pm 0.12$, in approximate agreement with $\beta = 1/2$. For the vortex part we obtained 0.24 ± 0.03 and a very clear power law $\kappa = 4$, indicating that it is the vortex part that is responsible for the scaling of the fundamental field correlator. In the three-component case, we obtained $\kappa = 5/2$ and $\beta = 0.53 \pm 0.1$ for the sound-wave phase correlator.

Finally we examined scaling in the 2+1D Sine-Gordon model. We found that its Boltzmann equation allows for scaling with $\beta = 1/6$, in addition to the well-known scaling with $\beta = 1/2$. This was corroborated by simulations of the Bessel-Gross-Pitaevskii model, being the non-relativistic limit of the Sine-Gordon model, where we found scaling with $\beta = 0.24 \pm 0.02$. On the one hand this provides evidence that the rather speculative derivation of the vortex/Sine-Gordon duality in 2+1D might indeed approximately hold.

On the other hand it gives rise to a possible explanation of the anomalously small β exponents in vortex-dominated systems. Admittedly there is still a slight discrepancy between the value of β extracted for $\langle\theta_{\mathrm{vo}}(\mathbf{k})\theta_{\mathrm{vo}}(-\mathbf{k})\rangle$ or the fundamental field correlator and $\beta = 1/6$, approximately of the size ~ 0.05 (though there is no discrepancy at all if one compares to the numerical computations of BGP theory). But in our opinion this is not too severe. First, there is a rather large error in the determination of such small exponents. And second, such a deviation could possibly be explained by subdominant contributions in the Boltzmann integral scaling with $\beta = 1/2$.

A plethora of new questions that could be addressed by further research arose during the course of this thesis project. First, since we laid much of the focus onto the Sine-Gordon model, the original subject of this thesis, the numerical study of phase correlators, was treated in far less depth than it actually would deserve. Especially we did not figure out the reason for the discrepancy between our numerical results and the analytical predictions from the low-energy effective theory of the phases in [8]. Various aspects of this effective theory could be studied in more detail numerically, e. g. the validity of assumptions about dispersion relations, effective couplings or the $1/N$ limit. Moreover, a treatment of the 3D case would be highly desirable.

Second, the question whether the 2+1D Sine-Gordon can indeed provide a description of vortex dynamics or whether the agreement of the β exponent is just a coincidence remains open. One way how one could possibly check that would be to calculate numerically classical vortex trajectories, preferably with the equation of motion arising from the action (5.26), or alternatively with some other effective equation of vortex dynamics or directly from GP simulations. These trajectories would then be inserted into (5.43) (which is still exact) and one would calculate the resulting dynamics of ϕ^μ with truncated Wigner simulations. From the results could then be inferred the range of validity of the assumptions we made.

Third, it is an intriguing question whether Sine-Gordon theory could provide also an effective description of the dynamics of topological defects in different systems, e. g. the 1+1D Luttinger liquid/XY model. Since the duality in 2D is known to be exact, one might suspect that this is also the case in 1+1D. The topological defects would be "time vortices" in this case. In d dimensions, the anomalous exponent from the Boltzmann equation is $\beta = 1/(2 + 2d)$, i. e. in one spatial dimension $\beta = 1/4$. This agrees with the value found for the 1+1D XY model from other approaches [39] and also the value extracted for the 1D spin system here. Whether this is a mere coincidence remains open, too. It has been demonstrated also for other systems with Euclidean space time that topological excitations can be described by the Euclidean Sine-Gordon model, e. g. for the 3D Abelian Higgs model [40] (where the topological excitations are small vortex loops) or the 4D XY model [41]. Here one can speculate too that after a Wick rotation the real time Sine-Gordon model describes the real time dynamics of these models in the presence of topological defects.

Finally, independent of the question whether it describes the dynamics of topological defects, the Sine-Gordon model far from equilibrium showed many interesting properties and deserves a much closer examination, regarding the 2PI/Boltzmann as well as the numerical treatment. In particular, it would be highly interesting to determine more precisely the regimes in which it exhibits scaling with the two different exponents $\beta = 1/2$ and $\beta = 1/(2 + 2d)$ that are found from the Boltzmann equation.

We hope that some of these questions can be resolved by future research.

A Conventions and units

Throughout this thesis we set

$$\hbar = 1. \tag{A.1}$$

However, the speed of light/sound c is not set to 1 in general but only in the last two chapters treating the relativistic Sine-Gordon model. I. e. anywhere else distances and time intervals are not measured in the same units. In this case we have e. g. the following relations:

$$\text{Energy} = \text{Time}^{-1} \tag{A.2}$$

$$\text{Momentum} = \text{Length}^{-1} \tag{A.3}$$

$$\text{Mass} = \text{Time} \times \text{Length}^{-2}. \tag{A.4}$$

If one sets $c = 1$ additionally, these collapse further to Energy = Momentum = Mass = Length^{-1}.
For the numerical calculations we work in numerical units. I. e. for the Gross-Pitaevskii model we set

$$a_s = 1 \tag{A.5}$$

$$m = \frac{1}{2} \tag{A.6}$$

with a_s the lattice spacing, which fixes the unit system. This means that e. g. $\rho_0 = 10^3$ has to be understood as $\rho_0 = 10^3 a_s^{-d}$. In the case of the BGP simulations it is sufficient to set $a_s = 1$ in order to fix the unit system since c has already been set to 1.
In general also momenta are given in numerical units, i. e. $Q = 0.5$ means $Q = 0.5 a_s^{-1}$. However, in the plots of correlation functions we use a different convention where we give a momentum as the number of the corresponding momentum mode on the lattice, i. e. here $k = 10$ has to be understood as $k = 10 \times 2\pi(a_s N)^{-1}$ with N the size of the lattice. In more than 1D, the function $f(k)$ has to be obtained from $f(\mathbf{k})$ by averaging over all angles. Since simulations are performed on a lattice, one must decide how to bin the different $\mathbf{k}$ vectors according to their absolute value. We chose a convention where we take as bin spacing the distance of the modes on the lattice, i. e. $2\pi(a_s N)^{-1}$. E. g. $\mathbf{k} = (1, 1) \times 2\pi(a_s N)^{-1}$ is binned into $k = 1 \times 2\pi(a_s N)^{-1}$ whereas $\mathbf{k} = (2, 1) \times 2\pi(a_s N)^{-1}$ is binned into $k = 2 \times 2\pi(a_s N)^{-1}$.

B Evaluation of the spatial integrals in the Luttinger liquid action

All we have to consider are the integrals

$$I_i^{xx} = \int d^2x \frac{(x-X_i)^2}{[(x-X_i)^2+(y-Y_i)^2]^2} \tag{B.1}$$

$$I_i^{yy} = \int d^2x \frac{(y-Y_i)^2}{[(x-X_i)^2+(y-Y_i)^2]^2} \tag{B.2}$$

$$I_i^{xy} = \int d^2x \frac{(x-X_i)(y-Y_i)}{[(x-X_i)^2+(y-Y_i)^2]^2} \tag{B.3}$$

and

$$J_{ij}^{xx} = \int d^2x \frac{(x-X_i)(x-X_j)}{[(x-X_i)^2+(y-Y_i)^2][(x-X_j)^2+(y-Y_j)^2]} \tag{B.4}$$

$$J_{ij}^{yy} = \int d^2x \frac{(y-Y_i)(y-Y_j)}{[(x-X_i)^2+(y-Y_i)^2][(x-X_j)^2+(y-Y_j)^2]} \tag{B.5}$$

$$J_{ij}^{xy} = \int d^2x \frac{(x-X_i)(y-Y_j)}{[(x-X_i)^2+(y-Y_i)^2][(x-X_j)^2+(y-Y_j)^2]} \tag{B.6}$$

with $i \neq j$. Of course the I_i^{yy} and J_{ij}^{yy} follow trivially from I_i^x and J_{ij}^{xx} by analogy. These integrals have already been calculated in [27] employing dimensional regularization. Here, however, we want to compute them regularizing by introducing a short distance cutoff a and a long distance cutoff L. In our opinion this is a more natural procedure in condensed matter systems where one can attribute a physical meaning to cutoffs.
Let us start with the I_i integrals. One can see from symmetry considerations that $I_i^{xy} = 0$ and $I_i^{xx} = I_i^{yy}$. Laying the origin at $\mathbf{X}_i$ and introducing polar coordinates we get

$$I_i^{xx} = \int_{-\pi}^{\pi} d\varphi \int_a^L \frac{dr}{r} \cos^2\varphi = \pi \ln \frac{L}{a}. \tag{B.7}$$

Now let us turn to the J_{ij}^{xx} integral, from which again by analogy will follow J_{ij}^{yy}. Again we lay the origin at $\mathbf{X}_i$. Furthermore we introduce the notation $\mathbf{X}_{ij} \equiv \mathbf{X}_j - \mathbf{X}_i$, $r_{ij} \equiv |\mathbf{X}_{ij}|$, $\widehat{X}_{ij} \equiv X_{ij}/r_{ij}$ and $\widehat{Y}_{ij} \equiv Y_{ij}/r_{ij}$. Now we introduce polar coordinates such that the angular coordinate φ is the angle between $\mathbf{x}$ and $\mathbf{X}_{ij}$. I. e. we write $x = r(\cos\varphi\widehat{X}_{ij} - \sin\varphi\widehat{Y}_{ij})$ and $y = r(\sin\varphi\widehat{X}_{ij} + \cos\varphi\widehat{Y}_{ij})$. With this we obtain

$$J_{ij}^{xx} = \int_{-\pi}^{\pi} d\varphi \int_0^L dr \frac{(\cos\varphi\widehat{X}_{ij} - \sin\varphi\widehat{Y}_{ij})(r\cos\varphi\widehat{X}_{ij} - r\sin\varphi\widehat{Y}_{ij} - r_{ij}\widehat{X}_{ij})}{r^2 + r_{ij}^2 - 2rr_{ij}\cos\varphi}. \tag{B.8}$$

To solve this, we need some integral formulas:

$$\int_{-\pi}^{\pi} d\varphi \frac{\cos^2\varphi}{1-t\cos\varphi} = \frac{2\pi}{t^2}\left(\frac{1}{\sqrt{1-t^2}}-1\right) \tag{B.9}$$

$$\int_{-\pi}^{\pi} d\varphi \frac{\sin^2\varphi}{1-t\cos\varphi} = \frac{2\pi}{t^2}\left(1-\sqrt{1-t^2}\right) \tag{B.10}$$

$$\int_{-\pi}^{\pi} d\varphi \frac{\cos\varphi}{1-t\cos\varphi} = \frac{2\pi}{t}\left(\frac{1}{\sqrt{1-t^2}}-1\right) \tag{B.11}$$

for $0 < t < 1$. All combinations containing only a single sine function in the numerator vanish due to symmetry. Applying these integral formulas as well as the identity

$$\sqrt{1-\left(\frac{2rr_{ij}}{r^2+r_{ij}^2}\right)^2} = \mathrm{sgn}(r-r_{ij})\frac{r^2-r_{ij}^2}{r^2+r_{ij}^2} \tag{B.12}$$

we obtain after some algebra:

$$\begin{aligned} J_{ij}^{xx} &= \pi \int_0^L dr \left[(\widehat{X}_{ij}^2+\widehat{Y}_{ij}^2)\frac{1}{r}\Theta(r-r_{ij}) + (-\widehat{X}_{ij}^2+\widehat{Y}_{ij}^2)\frac{r}{r_{ij}^2}\Theta(r_{ij}-r)\right] \\ &= -\pi\left[\ln\left(\frac{r_{ij}}{L}\right)+\widehat{X}_{ij}^2-\frac{1}{2}\right]. \end{aligned} \tag{B.13}$$

From analogy it follows that

$$J_{ij}^{yy} = -\pi\left[\ln\left(\frac{r_{ij}}{L}\right)+\widehat{Y}_{ij}^2-\frac{1}{2}\right]. \tag{B.14}$$

Therefore one finds for the spatial part of the Luttinger liquid action:

$$J_{ij}^{xx}+J_{ij}^{yy} = -2\pi\ln\left(\frac{r_{ij}}{L}\right). \tag{B.15}$$

The calculation for J_{ij}^{xy} is very similar and we only state the result here:

$$J_{ij}^{xy} = -\pi\widehat{X}_{ij}\widehat{Y}_{ij}. \tag{B.16}$$

(B.13), (B.14) and (B.16) agree with the results in [27] up to a difference in the constants stemming from the different regularization scheme.
Putting all the parts together after some algebra yields the effective action (5.26).

C Computation of the Wigner function

For the Wigner function of a coherent state $|\alpha\rangle$ we have

$$2\exp(-2|\alpha-\psi|^2). \tag{C.1}$$

Now we calculate the phase average of the Wigner function of the coherent state $\left|\sqrt{N_0}e^{i\theta_0}\right\rangle$, writing $\psi \equiv \sqrt{\rho}e^{i\theta}$:

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta_0 \exp\left(-2|\sqrt{N_0}e^{i\theta_0}-\sqrt{\rho}e^{i\theta}|^2\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta_0 \exp\left(-2\left(N_0+\rho-2\sqrt{N_0}\sqrt{\rho}\cos(\theta-\theta_0)\right)\right) \\
&= 2\exp\left(-2\left(N_0+\rho\right)\right) I_0\left(4\sqrt{N_0}\sqrt{\rho}\right)
\end{aligned} \tag{C.2}$$

where $I_0(x)$ denotes the modified Bessel function of zeroth order. For large values of x, the following approximation holds:

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}. \tag{C.3}$$

Therefore assuming $\sqrt{N_0}\sqrt{\rho} \gg 1$ we can write

$$2\exp\left(-2\left(N_0+\rho\right)\right) I_0\left(4\sqrt{N_0}\sqrt{\rho}\right) \approx \sqrt{\frac{1}{2\pi}}\frac{1}{(\rho N_0)^{1/4}}\exp\left(-2\left(\sqrt{N_0}-\sqrt{\rho}\right)^2\right). \tag{C.4}$$

Since the exponential suppresses all values of $\sqrt{\rho}$ which are not close to $\sqrt{N_0}$ we can additionally approximate $\rho \approx N_0$ in the denominator in order to arrive at the final result

$$\sqrt{\frac{1}{2\pi N_0}}\exp\left(-2\left(\sqrt{N_0}-\sqrt{\rho}\right)^2\right). \tag{C.5}$$

For the empty modes the calculation is even simpler. Here we just have to insert $\alpha = 0$ in (C.1) in order to arrive at

$$2\exp(-2\rho). \tag{C.6}$$

D Numerical evaluation of the Fourier transforms of $\sin(\rho/2)$ and $\cos(\rho/2)$

We show plots for the example parameters $\eta/4\pi = 1$, $m = 10$ (for other choices of parameters the results were very similar). For the discretization and the large distance cutoff we took $\epsilon = 0.025$ and $R = 10$. Figure (D.1) and (D.2) show contour plots of the ω-p plane. Here the approximation to expand the trigonometric seems to be quite accurate for cosine and sine as well. However, cross section plots (figure (D.3) and (D.4)) show that this is not entirely true for the sine. Whereas in the case of the cosine the expanded and full Fourier transform apparently agree quite accurately, this is not true for the sine where we have large deviations. The plot of the difference of expanded and full Fourier transform (figure D.5) reveals the problem: it shows a peak centered around the same position (namely $\sqrt{m^2+p^2}$) as $\mathcal{FT}\{\sin(\rho/2)\}$. Increasing η makes this peak higher and sharper, i. e. more δ-function like. Since it is a δ-function with approximately the same amplitude as that of $\rho(p)/2$ itself (as can be seen by considering the integral over them), $\mathcal{FT}\{\sin(\rho/2)\}$ will vanish unless all relevant frequencies are much smaller than $1/\eta$. This has to be assumed in order to enable scaling of the Boltzmann equation.

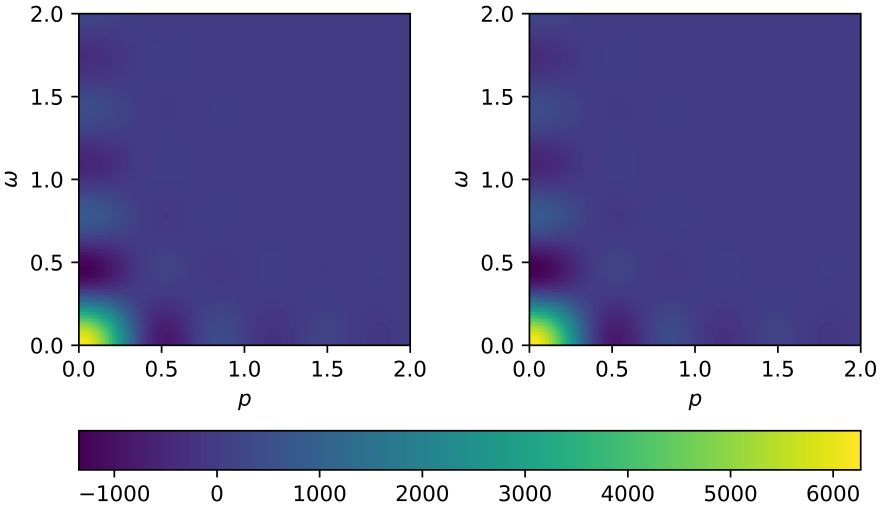

Figure D.1: Fourier transform of $\cos(\rho/2)$ (left) and 1 (right).

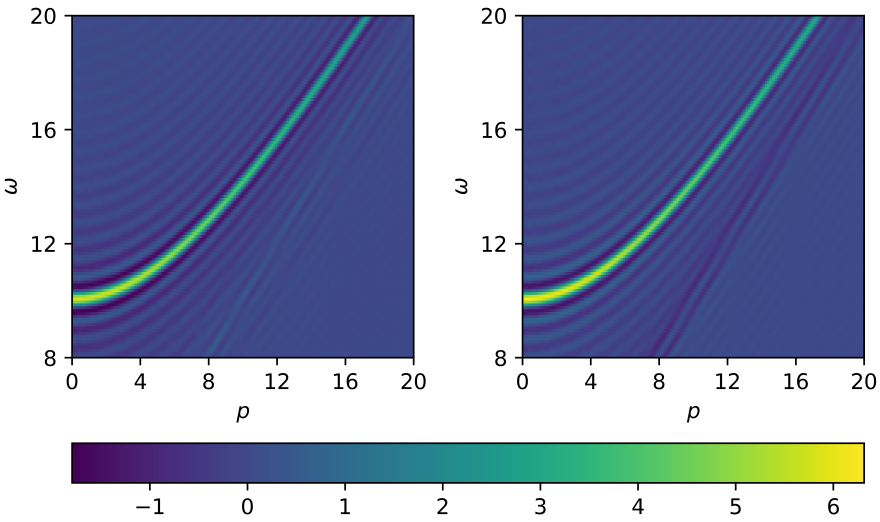

Figure D.2: Fourier transform of $\sin(\rho/2)$ (left) and $\rho/2$ (right).

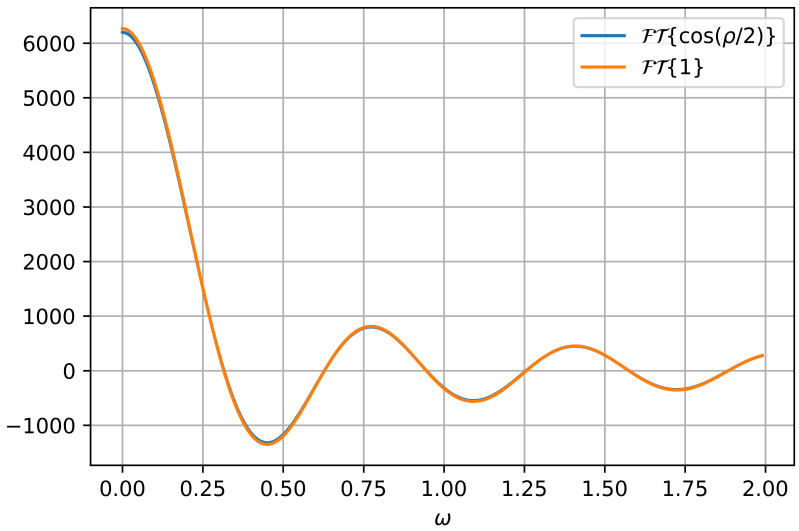

Figure D.3: Fourier transform of $\cos(\rho/2)$ and 1 for $p = 0$.

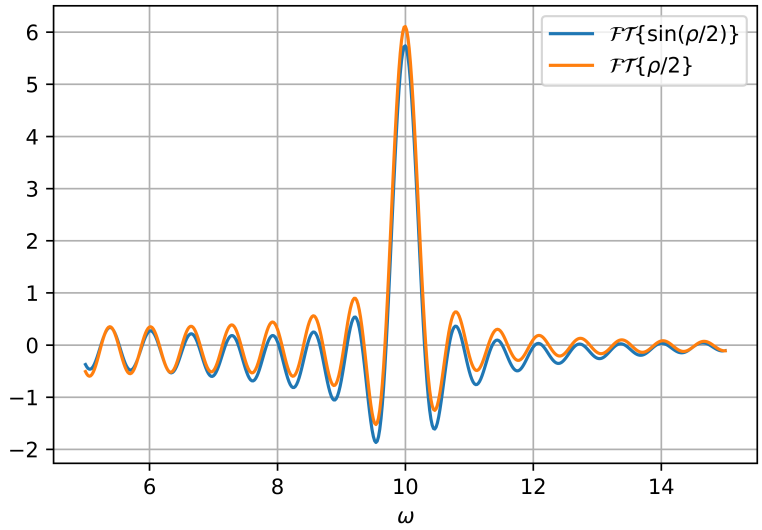

Figure D.4: Fourier transform of $\sin(\rho/2)$ and $\rho/2$ for $p = 0$.

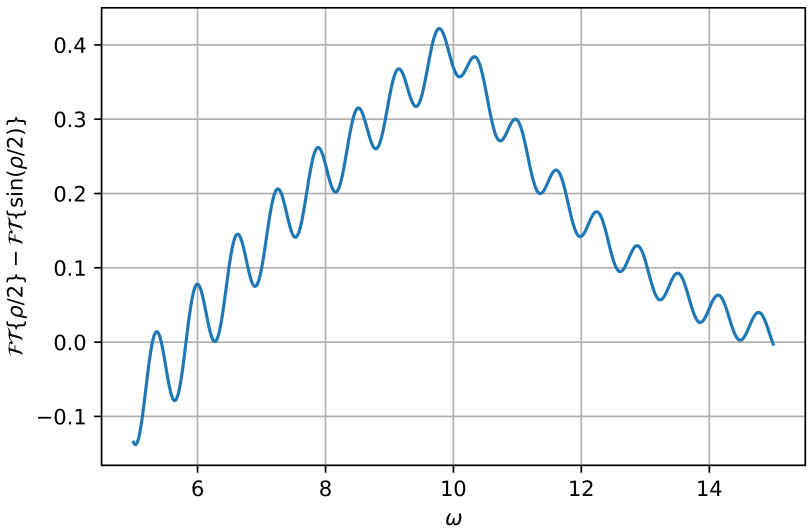

Figure D.5: Difference of the Fourier transforms of $\sin(\rho/2)$ and $\rho/2$ for $p = 0$.

E Additional correlator plots

Here we mainly show the rescaled versions of the correlator plots presented in the main text. Additionally figure (E.8) shows the full scalar phase correlator in the two-dimensional $U(3)$ system, including the part that in the main text is cut out.

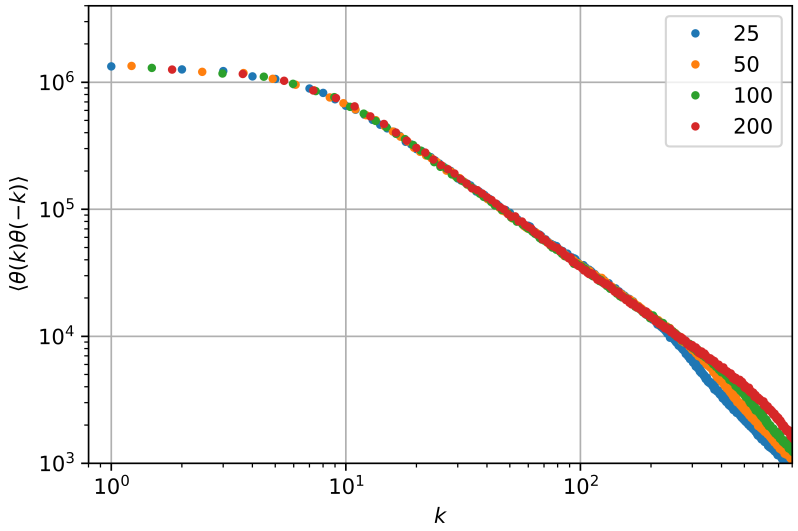

Figure E.1: The rescaled phase correlator in the 1D spin system calculated with the obvious definition.

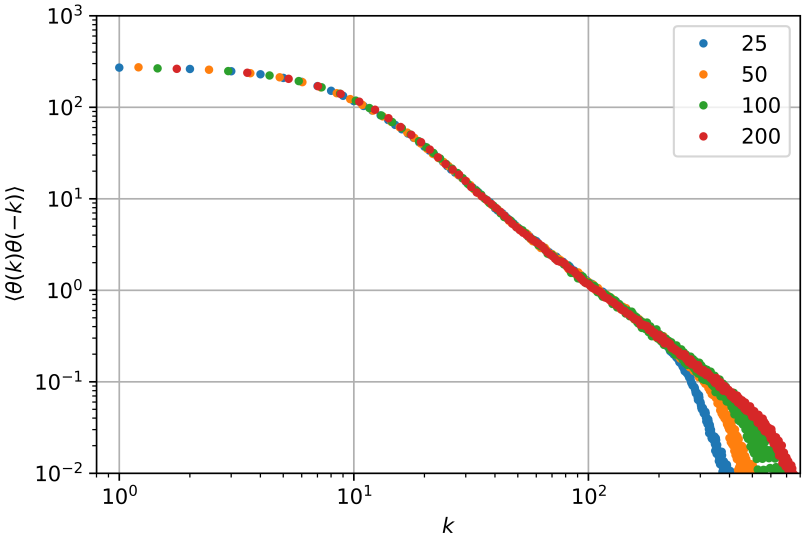

Figure E.2: The rescaled phase correlator in the 1D spin system calculated with the modification definition.

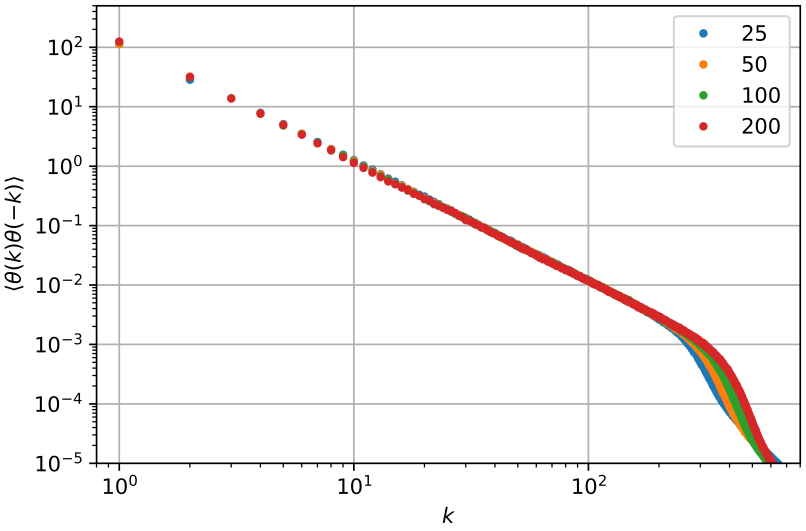

Figure E.3: The rescaled phase correlator in the 1D spin system calculated with the continuation definition.

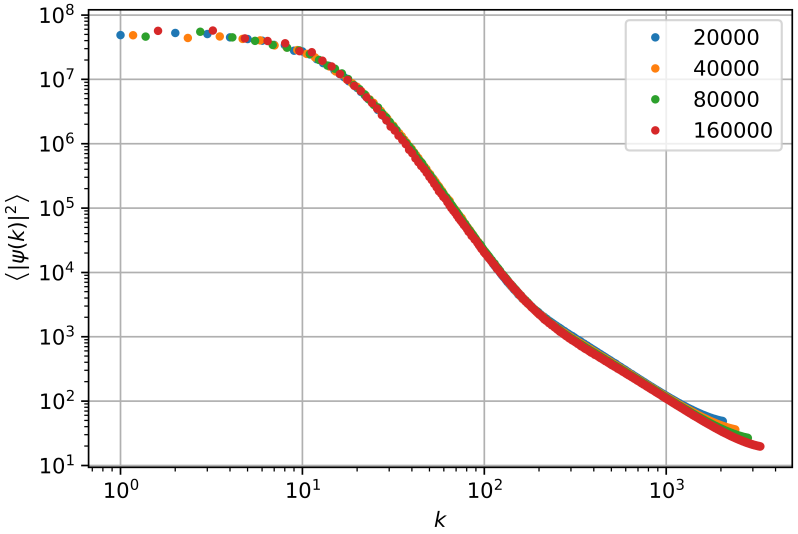

Figure E.4: Rescaled occupation number in the two-dimensional $U(1)$ system.

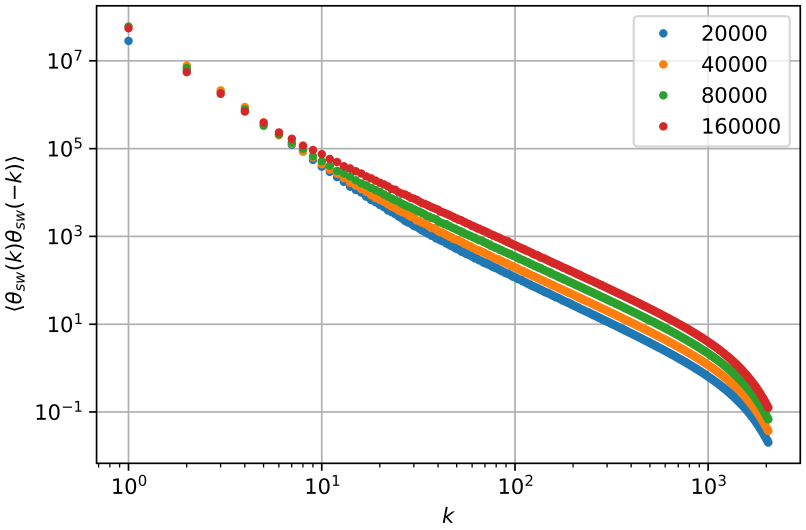

Figure E.5: Rescaled sound wave phase correlator in the two-dimensional $U(1)$ system.

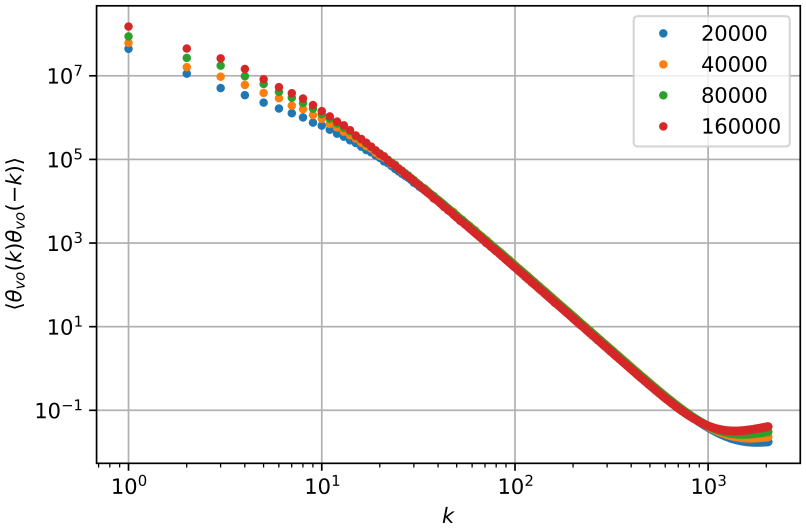

Figure E.6: Rescaled vortex phase correlator in the two-dimensional $U(1)$ system.

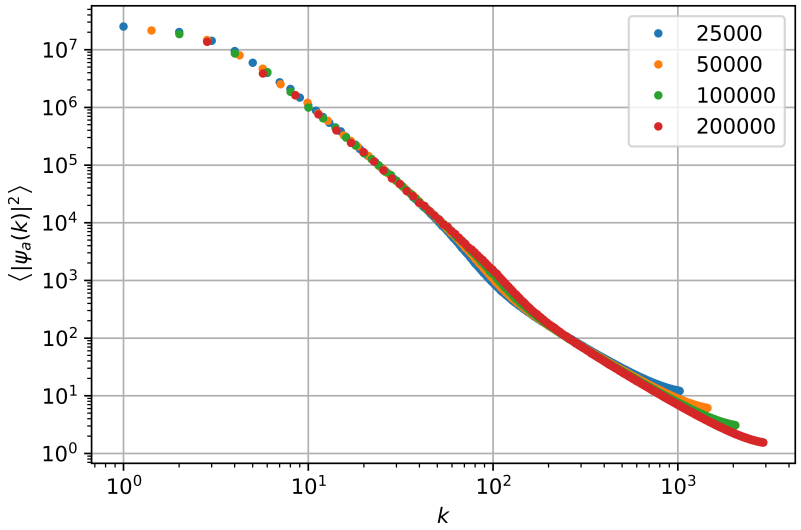

Figure E.7: Rescaled occupation number in the two-dimensional $U(3)$ system.

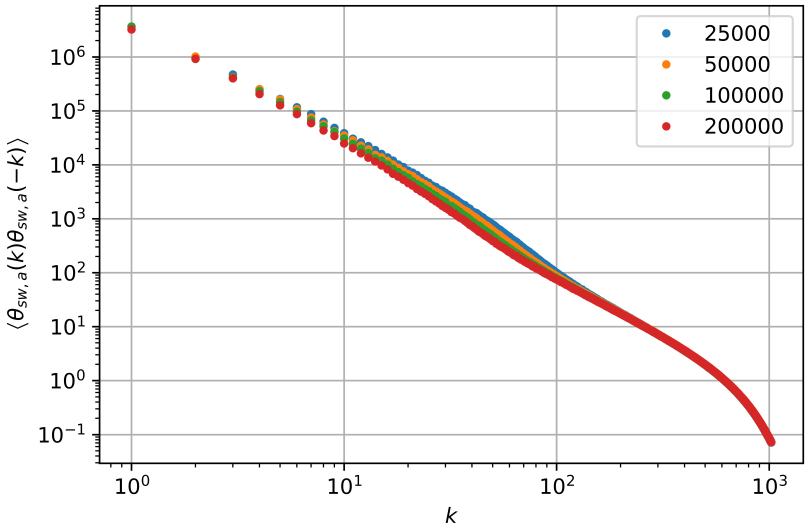

Figure E.8: Full sound wave phase correlator in the two-dimensional $U(3)$ system.

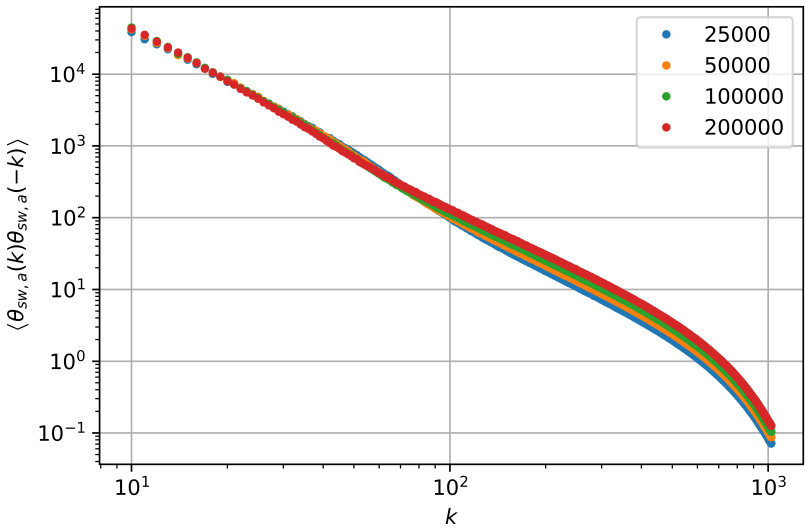

Figure E.9: Rescaled sound wave phase correlator in the two-dimensional $U(3)$ system.

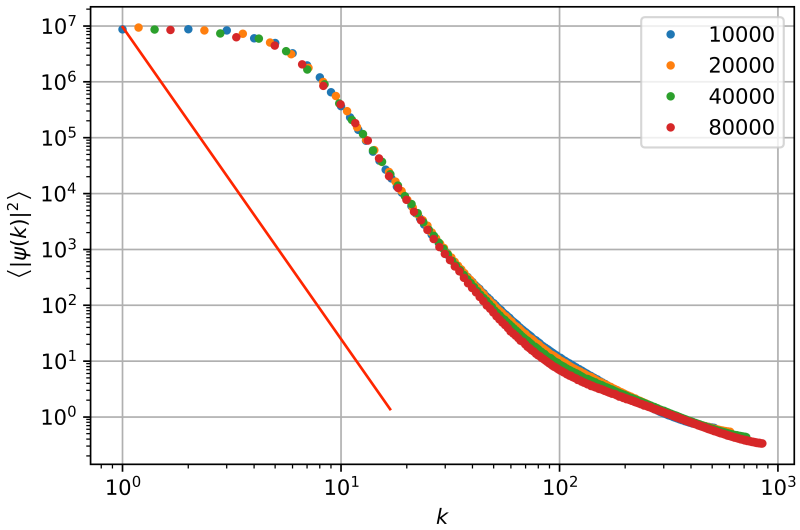

Figure E.10: Rescaled occupation number in the BGP system.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) .....