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Low-Energy Effective Theory of a Spin-1 Bose-Einstein Condensate in the Presence of the Quadratic Zeeman Effect

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Effektive Theorie eines Spin-1 Bose-Einstein-Kondensats in Gegenwart des Quadratischen Zeeman-Effekts bei niedrigen Energien:

Das wohlbekannte Goldstonetheorem besagt, dass das Anregungsspektrum von Systemen, deren lokaler Grundzustand durch spontane Symmetriebrechung gekennzeichnet ist, masselose Moden beinhaltet. In relativistischen Theorien entspricht die Anzahl dieser masselosen Quasiteilchen gerade der Anzahl an gebrochenen Erzeugern der Symmetriegruppe. In nicht-relativistischen Systemen ist dies im Allgemeinen jedoch nicht der Fall. Stattdessen gilt, dass die Anzahl der gebrochenen Erzeuger gleich der Zahl derjenigen Feldvariablen ist, die ohne Massenterm in die Lagrangefunktion eingehen. Die Quantisierung der effektiven Feldtheorie involviert die Eliminierung von Zwangsbedingungen, welche beim Wechsel zwischen Lagrange-Formalismus und Hamilton-Formalismus auftreten. Daraus ergibt sich eine Reduzierung an Freiheitsgraden, weshalb die resultierende Zahl der masselosen Moden im Allgemeinen kleiner als die Zahl der gebrochenen Erzeuger ist. Die zugrundeliegende Methodik der Behandlung solcher Systeme wird präsentiert und auf ein Spin-1 Bosegas in der *broken axissymmetry* Phase angewandt. Mithilfe der resultierenden effektiven Theorie für niedrige Energien wird die Dynamik des Systems in der Nähe eines nichtthermischen Fixpunkts störungstheoretisch untersucht. Unter der Annahme einer selbstähnlichen Zeitentwicklung der statistischen Funktion werden die entsprechenden Skalierungsexponenten berechnet und eine gute Übereinstimmung mit experimentellen Resultaten festgestellt.

Low-Energy Effective Theory of a Spin-1 Bose-Einstein Condensate in the Presence of The Quadratic Zeeman Effect:

A well-known theorem due to Goldstone states that there are massless excitations in the vicinity of a local ground state of any system which exhibits spontaneous symmetry breaking. In a relativistic context the number of these excitation modes equals the number of broken symmetry generators. However, in non-relativistic systems this is generally not the case. Instead, one can only equate the number of fields whose mass matrix vanishes with the number of broken generators. The quantization of the low-energy effective Lagrangian involves the treatment of constraints which arise when transitioning from the Lagrangian formulation to the Hamiltonian. This results in a reduction of the number of degrees of freedom and therefore the number of massless modes is in general lower than the number of broken generators. The principles and techniques involved in treating such systems are discussed and applied to a spin-1 bose gas in the broken-axissymmetry phase. The resulting low-energy effective theory is then used for a perturbative analysis of its dynamics in the vicinity of a non-thermal fixed point. Under the assumption of a self-similar time evolution of the statistical functions we calculate the corresponding scaling exponents and find them to be in good agreement with experimentally observed values.

Contents

1 Introduction

In order to theoretically study the behaviour of a system at very low temperatures, one would like to work within a low energy effective theory instead of taking all of its degrees of freedom into account. As we know from the general scheme of renormalization, it is not necessary to explicitly include degrees of freedom whose energy spectrum lies above the energy scale of interest and excluding them usually simplifies the description. For this purpose one needs to identify the various modes of excitation and their spectrum. One possibility is to start with a Hamiltonian that correctly describes the system at low temperature or even beyond. Diagonalizing the quadratic part gives the modes and spectra and the left-over terms constitute interactions between these quasiparticles. By formulating the theory within the path integral formalism one can then proceed by integrating out those modes which have a gapped spectrum, provided that the gap is larger than the typical energy of excitations in the system for a given experimental setup. This results in a modification of the interaction terms for the remaining degrees of freedom but may also contribute to the quadratic part of the remaining effective Hamiltonian and therefore to the spectrum of the low energy effective theory. When starting from a (classical) Lagrangian description, there is also the task of constructing the quantum Hamiltonian in the first place. While often straightforward, there can be complications arising at this stage. Sometimes the naive construction of the Hamiltonian and the corresponding phase space by defining the canonical momenta and performing the usual Legendre transformation does not lead to a correct Hamiltonian description. This is essentially due to the fact that the Lagrangian might not be a convex function of the time derivatives of the fields. In this case the relation between momenta and time derivatives of fields is not invertible and so it is not possible to express the latter in terms of the former. However, it is always possible to formally write down the usual Legendre transformation of the Lagrangian and obtain a Hamiltonian which is a function of fields and momenta only. This is because one either has an invertible relation, which allows for a re-expression of the time derivatives in terms of momenta, or one has a cancellation of the time derivative terms. In the second case this naive Hamiltonian does not correctly describe the dynamics of the system however because here some of the Euler-Lagrange-equations impose explicit constraints on the phase space by relating fields and momenta instead of merely providing equations of motion. This means that the real phase space of the system is a symplectic submanifold of the naive phase space. The usual transition to the Hamiltonian formalism with its equations of motion relies on the phase space being the full naive phase space not subject to any constraints and so this transition has to be modified for cases with constraints. While being an issue already at the classical level, this is especially important when quantizing the system since the commuation relations of fields and momenta will generally be affected by constraints. By a certain choice of canonical transformation this can be circumvented, however.

In order to avoid involved calculations, it might be preferable to construct a low

energy effective theory at the Lagrangian level and the quantize it instead of the other way around. If the transition from the full Lagrangian to the effective one is computationally simpler than integrating out all of the gapped modes in the final Hamiltonian, then one might have a more efficient method at hand. In order to see how this works, one should firstly note that for many systems there a several possible, qualitatively different ground states, i.e. local minima of its energy spectrum. Typically some of the fields appearing in the original Lagrangian acquire a non-vanishing expection value in these ground states and these values serve as order parameters defining the particular ground state or "phase" of the system. These non-vanishing order parameters imply that the set of field ground state values typically is not invariant under all of the symmetries of the Lagrangian. This phenomenon is referred to as spontaneous symmetry breaking and simply means that the system's state does not respect the system's dynamical symmetry. A standard example for spontaneous symmetry breaking is the ground state of a ferromagnet (at vanishing temperature) since the Lagrangian is invariant under uniform O(3) rotations of all of the spins but in the ferromagnetic phase all of the spins are aligned in order to minimize the energy. This configuration of spins is clearly not invariant under general O(3) rotations. It is however under O(2) rotations about the axis of alignment. This goes to show that the original symmetry need not be broken completely but generally there will be an unbroken subgroup of the original symmetry group remaining under which the ground state remains invariant. In the example of the ferromagnetic phase the original group O(3) is broken to O(2). A theorem which was proven by Goldstone [20] states that whenever there is a spontaneously broken symmetry there are gapless modes appearing in the spectrum of excitations around the given ground state. It also states that the number of these modes is just the number of broken symmetry generators, that is the dimensionality of the original full symmetry group minus the dimensionality of the remaining unbroken symmetry group. This version of the theorem is only correct in the case of a relativistic QFT, however. This is precisely due to the fact that the Lagrangian in that case is of second order in the time derivatives and therefore convex, so there are no constraints arising at the transition to the Hamiltonian and each of the fields associated with a broken symmetry generator becomes its own massless degree of freedom. In the non-relativistic case one can and does have Lagrangians of first order in the time derivatives of fields and so constraints do arise and reduce the number of actual degrees of freedom in the quantized Hamiltonian. There is indeed a general method for constructing the effective Lagrangian in each phase of the system by just using the symmetry breaking pattern. This method was developed by Coleman, Wess and Zumino [5],[3] and might remain valid for non-relativistic systems since the complications involved here only occur at the stage of quantization. The central claim being that the low energy effective Lagrangian is the so called non-linear σ-model with the target space G/H, where G is the dynamical symmetry group of the system and H the subgroup of G that remains unbroken by the ground state. While in the relativistic case there is a clear separation between massive fields and modes on one side and Goldstone-fields and massless modes on the other side, in the non-relativistic case this is a priori not clear due to the non-invertible correspondence between the classical Lagrangian and the quantized Hamiltonian.

1.1 Motivating Non-Linear Realizations

1.1.1 The Non-Linear σ-Model

The origin of the study of non-linear realizations in physics is a paper by Gell-Mann and Levy [7] in which they consider an O(4) symmetric Lagrangian of four fundamental fields $\Psi = (\vec{\pi}, \sigma)$, where the three π-fields describe particles called pions and the σ field would be associated with a hypothetical σ-particle. The symmetry group O(4) acts via the vector-representation on the vector of fields and therefore linearly:

$$O(4) \ni R: \quad \Psi_i \mapsto \sum_{j=1}^{4} R_{ij} \Psi_j \tag{1.1.1}$$

This is the most common case of group actions appearing in physics and the mathematics of representation theory describes and categorizes these linear actions in detail. The problem with this so-called linear σ-model is that there is no evidence for the σ-particle and yet it seemed to describe the pion-fields and their symmetry correctly. In order to remove the σ as and independent degree of freedom one could make it a function of the pion fields $\sigma(\vec{\pi})$. Since one wants to preserve the transformation properties of the vector $(\vec{\pi}, \sigma)$ under O(4) and the norm of this vector is invariant under these rotations, we conclude that the condition $\vec{\pi}^2 + \sigma(\vec{\pi})^2 = c^2$ with c constant must be met. Therefore we have $\sigma(\vec{\pi}) = \sqrt{c^2 - \vec{\pi}^2}$ and the transformation of the pion-fields under O(4) rotations follows:

$$O(4) \ni R: \quad \pi_a \mapsto \sum_{b=1}^{3} R_{ab} \pi_b + R_{a0} \sqrt{c^2 - \vec{\pi}^2} \tag{1.1.2}$$

This is clearly a non-linear transformation. Group actions of this kind are referred to as non-linear realizations of the symmetry group. Instead of eliminating the σ-degree of freedom completely by fixing the norm of the field vector, one could keep it but parameterize the vector space by using σ as a radius and define the π fields as coordinates on the sphere of constant radius σ. The non-linearity of this change of variables will then show up as a non-linearity in the transformation property of the π-fields under O(4). While this kind of construction of non-linear realizations from linear representations is rather typical, one need not make any reference to linear models in the first place. That is, non-linear realizations can be studied purely on the basis of group theory and differential geometry alone and even have to be studied in this generality in order to proof that the Lagrangian for a system that exhibits spontaneous symmetry breaking can be formulated in the fashion we will explore in the next section.

The pion model is the first and maybe most famous model of this kind but since then it has been generalized and applied to many systems. The term 'non-linear σ model' has been adopted for this entire class of models which are based on the breaking of a symmetry group to one of its subgroups. In order to see why the pion model is of this type we note that by choosing a particular point on a sphere of fixed radius σ as the origin of our parameterization we break the O(4) symmetry,

since this particular point is not fixed under general rotations. It is, however, fixed under rotations about the axis defined by the origin and this particular point on the sphere and so there is an O(3) symmetry, with O(3) being a proper subgroup of O(4). The sphere is isomorphic to the quotient space O(4)/O(3) and so the π-fields are just coordinates on this quotient space. This is in fact the general situation for systems in which the original symmetry group G is spontaneously broken to a subgroup H. A parameterization of the manifold G/H corresponds to degrees of freedom which transform non-linearly under G.

2 Spontaneously Broken Symmetries

2.1 General Construction [1]

The most general formalism for dealing with spontaneously broken symmetries does not make reference to an original Lagrangian which is written in terms of what we shall call the fundamental fields. However, it requires the use of some more abstract differential geometry and as it turns out, one can show that the identical result can always be obtained by starting with a Lagrangian written in terms of some fundamental fields and following the procedure that we will lay out in this section. By definition, these fundamental fields are acted upon by the symmetry group through a linear representation. This - in fact non-restrictive - simplification allows us to avoid the introduction of geometrical structures like principal bundles and to use only basic linear algebra. Although it is often convenient to formulate a real Lagrangian in terms of complex fields and the linear representation of a symmetry group in terms of a complex matrix algebra, this can always be rewritten in terms of real fields and a real matrix representation. Since each real field counts as a separate field we will always do just that or assume that it has been done already.

Let G denote the full inner-symmetry Lie-group of the Lagrangian, which we will assume to be compact and $\psi = (\psi_n)_{n=1,\dots,N}$ the set of fundamental fields. A group element $g \in G$ then acts on ψ through a matrix $[g]$:

$$(g\psi)_n = \sum_{m=1}^{N} [g]_{nm}\psi_m \tag{2.1.1}$$

We will denote the ground state expectation values of the fundamental field components by $\langle\psi_n\rangle_{GS}$. These values differ depending on the particular phase that the system is in and serve to define and distinguish these phases. The absence of any space-time arguments in these equations is due to the fact that we will only consider the case where space-time symmetries are not spontaneously broken, whether they be the relativistic Poincare group or the non-relativistic Gallilei group. This means in particular that the ground state expectation values $\langle\psi_n(x)\rangle_{GS}$ are invariant under translations and therefore do not depend on the coordinates x. That the symmetry group G is broken to the subgroup H means that there is a sub-Lie-group $H \subset G$ such that the action of any $h \in H$ on the set of order parameters $\langle\psi_n\rangle_{GS}$ leaves them invariant:

$$\sum_{m=1}^{N} [h]_{nm} \langle\psi_m\rangle_{GS} = \langle\psi_n\rangle_{GS} \tag{2.1.2}$$

[1] For the original references see [5] and [3]. The structure and presentation of this chapter is largely based on [20], supplemented with more detailed calculations.

Group elements that are not in H will map the original set of order parameters to a different one. Since G is a symmetry of the theory, all of them correspond to a ground state of equal energy and together they form a submanifold in field space of degenerate ground states. That is, any set of order parameters generated by a mapping with $g \in G$ represents an different but equivalent ground state and we may choose any of them as the starting point of our construction without affecting the resulting low-energy effective theory. Next, one reparameterizes the fundamental fields in terms of a new set of fields that is suited to the symmetry breaking pattern. This is done by first defining the set of group generators $(w_\lambda)_{\lambda=1,...,\dim(G)}$ which satisfy

$$G = \{\exp[\mathrm{i}\, z_\lambda w_\lambda] \,|\, z \in \mathbb{R}^{\dim(G)}\}. \tag{2.1.3}$$

The appearance of the prefactor i is by pure convention and one should keep in mind that the generators w are purely imaginery matrices and hence we have a real representation. Since G is a compact group, every field configuration can be written as a certain group element γ acting on a special set of fields $\tilde{\psi}$:

$$\psi_\mathrm{n} = \sum_{\mathrm{m}=1}^{\mathrm{N}} [\gamma]_{\mathrm{nm}} \tilde{\psi}_\mathrm{m} \tag{2.1.4}$$

Since ψ is spacetime dependent, so are γ and $\tilde{\psi}$ and one should note that the number of field components in $\tilde{\psi}$ is less than N, with the missing degrees of freedom hiding in the group element γ. The essential idea is that the degrees of freedom in γ parameterize the submanifold of groundstate and therefore a surface of constant potential. This should result in massless degrees of freedom because there cannot be any terms in L which depend on the value of these fields only. Instead every term depending on such fields has to involve their spacetime derivatives and will be a kinetic term. In contrast the $\tilde{\psi}$ are orthogonal to this submanifold an will have mass terms. More precisely they are defined by the following condition:

$$\forall \lambda = 1, ..., \dim G: \qquad \sum_{\mathrm{nm}} \tilde{\psi}_\mathrm{n} [w_\lambda]_{\mathrm{nm}} \langle \psi_\mathrm{m} \rangle_{\mathrm{GS}} = 0 \tag{2.1.5}$$

Because any group element of H annihilates the vector $\langle \psi \rangle_{\mathrm{GS}}$, there are only dim($G$) - dim($H$) conditions present here which result in a set of N-(dim(G)-dim(H)) solutions for $\tilde{\psi}$. The general $\tilde{\psi}$ will then be a linear combination of these solutions and so there are N-(dim(G)-dim(H)) degrees of freedom constituting this field. The choice of γ and $\tilde{\psi}$ in (2.1.4) is not unique. In fact one can multiply γ from the right by any element of $h \in H$ and define the group element $\gamma' = \gamma h$, as well as multiply $\tilde{\psi}$ from the left by the inverse element h^{-1} and define $\tilde{\psi}' = h^{-1}\tilde{\psi}$. If $\tilde{\psi}$ satisfies (2.1.5) then so does $\tilde{\psi}'$ and therefore these quantities are defined only up to transformations with H. This shows that the number of degrees of freedom that have been absorbed into the group element γ is in fact not dim(G) but dim(G)-dim(H). To be precise, these degrees of freedom constitute a parameterizaion of the coset space G/H (where dim(G/H) = dim(G)-dim(H)) and can be thought of as coordi-

nates on the submanifold of degenerate ground states in field configuration space. In order to choose a representative group element γ we first split the Lie algebra $g = \mathrm{span}((w_\lambda)_{\lambda=1,\ldots,\dim(G)})$ of G into the closed sub-Lie algebra h that generates H and a suitable complement. Let $(t_\mathrm{i})_{\mathrm{i}=1,\ldots,\dim(H)}$ be a basis of h and $(x_\mathrm{a})_{\mathrm{a}=1,\ldots,(\dim(G)-\dim(H))}$ such that the combined set forms a basis of g. We adopt the convention that indices i,j,k... refer to t's and indices a,b,c... refer to x's. With H being a sub Lie group we have

$$[t_\mathrm{i}, t_\mathrm{j}] = \mathrm{i} \sum_\mathrm{k} C_\mathrm{ijk} t_\mathrm{k} \tag{2.1.6}$$

The set $(x_\mathrm{a})_{\mathrm{a}=1,\ldots,(\dim(G)-\dim(H))}$ can alway be chosen such that

$$[t_\mathrm{i}, x_\mathrm{a}] = \mathrm{i} \sum_\mathrm{b} C_\mathrm{iab} x_\mathrm{b} \quad , \tag{2.1.7}$$

$$[x_\mathrm{a}, x_\mathrm{b}] = \mathrm{i} \sum_\mathrm{i} C_\mathrm{abi} t_\mathrm{i} + \mathrm{i} \sum_\mathrm{c} C_\mathrm{abc} x_\mathrm{c} \quad . \tag{2.1.8}$$

The simplification here being that all structure constants C_ija vanish. This means that the x generators will transform linearly under the adjoint map with respect to elements of H. An arbitrary group element $g \in G$ can now be written in the form

$$g = \exp\left[\mathrm{i} \sum_\mathrm{a} \pi_\mathrm{a} x_\mathrm{a}\right] \exp\left[\mathrm{i} \sum_\mathrm{i} \theta_\mathrm{i} t_\mathrm{i}\right] \quad . \tag{2.1.9}$$

The second factor, being an element of H, can be dropped to select as a representative of $\bar{g} \in G/H$ the group element

$$\gamma_{\bar{g}} = \exp\left[\mathrm{i} \sum_{\mathrm{a}=1}^{\dim(G/H)} \pi_\mathrm{a} x_\mathrm{a}\right] \quad . \tag{2.1.10}$$

The π_a are called Goldstone fields and complement the set $\tilde{\psi}$ to provide a complete reparameterization of the fundamental fields ψ. In order to see the kind of terms one obtains for a Lagrangian that is expressed in terms of π's and $\tilde{\psi}$'s we can imagine ourselves explicitly inserting this reparameterization of fields into the original Lagrangian density $\mathcal{L}$. Since we assume the Lagrangian to be symmetric under global G transformations and also local, any term not containing spacetime derivatives will be invariant even under local transformations and so the spacetime dependence of the π fields will not alter the fact that $\gamma_{\bar{g}}$ cancels out for such terms. For any term involving $\partial_\mu \psi$ we need to take into account that the derivative acts on $\gamma_{\bar{g}}$ as well as on $\tilde{\psi}$:

$$\partial_\mu \psi = \gamma_{\bar{g}} [\partial_\mu \tilde{\psi} + (\gamma_{\bar{g}}^{-1} \partial_\mu \gamma_{\bar{g}}) \tilde{\psi}] \tag{2.1.11}$$

We should note once more, that for global transformations $\partial_\mu \psi \to g \partial_\mu \psi$ and g has

to cancel out in each term. Even though $\gamma_{\bar{g}}$ is spacetime dependent in (2.1.11), the locality of $\mathcal{L}$ ensures that the overall factor $\gamma_{\bar{g}}$ in front of the right hand side will cancel out in each term. This means that the appearance of the Goldstone fields is contained in $\gamma_{\bar{g}}^{-1}\partial_\mu\gamma_{\bar{g}}$ and only involves terms which are at least of first order in spacetime derivatives of the π's. The expression $\gamma_{\bar{g}}^{-1}\partial_\mu\gamma_{\bar{g}}$ can be evaluated using Duhamel's formula for the derivative of the exponential map:

$$\partial\left(\mathrm{e}^{X(\cdot)}\right) = \mathrm{e}^{X(\cdot)}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}}{(k+1)!}(ad_{X(\cdot)})^{\mathrm{k}}(\partial X(\cdot)) \tag{2.1.12}$$

Here $X : \mathbb{R} \to g$ describes a path in the Lie algebra of G and ad denotes the adjoint action of the Lie algebra on itself, i.e. $ad_X(Y) = [X, Y]$. In our case we have $X = \mathrm{i}\sum_{\mathrm{a}} \pi_{\mathrm{a}}(\cdot)x_{\mathrm{a}}$ and so

$$\gamma_{\bar{g}}^{-1}\partial_\mu\gamma_{\bar{g}} = \mathrm{i}\sum_{\mathrm{a}}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}}{(k+1)!}(ad_{(\mathrm{i}\sum_{\mathrm{b}}\pi_{\mathrm{b}}x_{\mathrm{b}})})^{\mathrm{k}}(x_{\mathrm{a}})\partial_\mu\pi_{\mathrm{a}} \tag{2.1.13}$$

$$= \sum_{\mathrm{b}}\left[\mathrm{i}\sum_{\mathrm{a}} x_{\mathrm{a}}\mathrm{D}_{\mathrm{ab}}(\pi) + \mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\mathrm{E}_{\mathrm{ib}}(\pi)\right]\partial_\mu\pi_{\mathrm{b}} \tag{2.1.14}$$

In the second line we made use of the fact that

$$\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}}{(k+1)!}(ad_{(\mathrm{i}\sum_{\mathrm{b}}\pi_{\mathrm{b}}x_{\mathrm{b}})})^{\mathrm{k}}(x_{\mathrm{a}})$$

is a linear combination of the generators with π-dependent coefficients which we named D and E. We can calculate these coefficients as a power series in π by evaluating the above expression to arbitrary order in the series over k:

$$\begin{aligned}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}}{(k+1)!}(ad_{(\mathrm{i}\sum_{\mathrm{b}}\pi_{\mathrm{b}}x_{\mathrm{b}})})^{\mathrm{k}}(x_{\mathrm{a}}) &= x_{\mathrm{a}} - \frac{1}{2}\left[\mathrm{i}\sum_{\mathrm{b}}\pi_{\mathrm{b}}x_{\mathrm{b}}, \mathrm{i}x_{\mathrm{a}}\right] + \frac{1}{6}\left[\mathrm{i}\sum_{\mathrm{c}}\pi_{\mathrm{c}}x_{\mathrm{c}}, \left[\mathrm{i}\sum_{\mathrm{b}}\pi_{\mathrm{b}}x_{\mathrm{b}}, \mathrm{i}x_{\mathrm{a}}\right]\right] + \dots \\ &= x_{\mathrm{a}} + \frac{1}{2}\sum_{\mathrm{b}}\pi_{\mathrm{b}}\left(\sum_{\mathrm{c}} C_{\mathrm{bac}}x_{\mathrm{c}} + \sum_{\mathrm{i}} C_{\mathrm{bai}}t_{\mathrm{i}}\right) \\ &\quad + \frac{1}{6}\sum_{\mathrm{b,c}}\pi_{\mathrm{b}}\pi_{\mathrm{c}}\left(\sum_{\mathrm{e}}\left(C_{\mathrm{bad}}C_{\mathrm{cde}} + C_{\mathrm{bai}}C_{\mathrm{cie}}\right)x_{\mathrm{e}} + \sum_{\mathrm{j}} C_{\mathrm{bad}}C_{\mathrm{cdj}}t_{\mathrm{j}}\right) + \dots\end{aligned} \tag{2.1.15}$$

By comparing the two lines in (2.1.13) one finds

$$D_{ab} = \delta_{ab} + \frac{1}{2}\sum_c C_{cba}\pi_c + \frac{1}{6}\sum_{c,d}\left(\sum_e C_{dbe}C_{cea} + \sum_i C_{dbi}C_{cia}\right)\pi_c\pi_d + \dots \tag{2.1.16}$$

$$E_{ib} = \frac{1}{2}\sum_c C_{cbi}\pi_c + \frac{1}{6}\sum_e C_{cbe}C_{dei}\pi_c\pi_d + \dots \tag{2.1.17}$$

Since we know now that the Goldstone fields appear in the Lagrangian only through these coefficients and we would eventually like to use the symmetries of $\mathcal{L}$ in order to construct it without explicitly reparameterizing some previously given Lagrangian, it makes sense to study the transformation behaviour of the new fields and of D and E in particular. This can be derived from the linear group action on the fundamental fields:

$$\psi \to g\psi = g\gamma(\pi)\tilde{\psi} \tag{2.1.18}$$

With $g\gamma(\pi) \in G$ we know that there exists a representative $\gamma(\pi')$ of $\overline{g\gamma(\pi)}$ and a $h(\pi, g) \in H$ such that

$$g\gamma(\pi) = \gamma(\pi')\, h(\pi, g) \quad . \tag{2.1.19}$$

This implicitly defines the transformation of π under G and by comparing (2.1.18) with (2.1.4) we find the relation

$$\tilde{\psi}' = h(\pi, g)\tilde{\psi} \quad . \tag{2.1.20}$$

Even though the π's transform in a complicated manner, the expression $\gamma^{-1}\partial_\mu\gamma$ transforms rather simply:

$$\begin{aligned} \gamma^{-1}(\pi)\partial_\mu\gamma(\pi) &= [g\gamma(\pi)]^{-1}\partial_\mu[g\gamma(\pi)] \\ &= h^{-1}(\pi, g)\gamma^{-1}(\pi')\partial_\mu[\gamma(\pi')h(\pi, g)] \\ &= h^{-1}(\pi, g)\,[\gamma^{-1}(\pi')\partial_\mu\gamma(\pi')]\,h(\pi, g) + h^{-1}(\pi, g)\partial_\mu h(\pi, g) \\ \Rightarrow \quad \gamma^{-1}(\pi')\partial_\mu\gamma(\pi') &= h(\pi, g)\,[\gamma^{-1}(\pi)\partial_\mu\gamma(\pi)]\,h^{-1}(\pi, g) - [\partial_\mu h(\pi, g)]h^{-1}(\pi, g) \end{aligned} \tag{2.1.21}$$

Here we used eq. (2.1.19) in line two and the product rule in line three. At this point the special choice of a Lie algebra basis that was made in (2.1.7)-(2.1.8) pays off because C_{ija} vanishing ensures that the x transform linearly under H in the sense that

$$\mathrm{Ad}_h(x_a) = hx_ah^{-1} = \sum_b \mathcal{D}_{ba}(h)x_b \quad . \tag{2.1.22}$$

This can be seen by writing

$$h = \exp\left[\mathrm{i}\sum_{\mathrm{j}} \theta_{\mathrm{j}} t_{\mathrm{j}}\right] \tag{2.1.23}$$

and using the identity $\mathrm{Ad}_{\exp[X]} = \exp[\mathrm{ad}_X]$ together with $C_{\mathrm{ija}} = 0$, which ensures that

$$\exp\left[\mathrm{ad}_{(\mathrm{i}\sum_{\mathrm{j}} \theta_{\mathrm{j}} t_{\mathrm{j}})}\right](x_{\mathrm{a}}) \in \mathrm{span}(x) \quad . \tag{2.1.24}$$

It also follows that the partial derivative of $\mathcal{D}$ with respect to θ_{i} is

$$\partial_{\mathrm{i}} \mathcal{D}_{\mathrm{ab}} = \sum_{\mathrm{c}} \mathcal{D}_{\mathrm{ac}} C_{\mathrm{icb}} \quad . \tag{2.1.25}$$

This relation will later become useful because it allows us to determine the structure of the coefficients in the Lagrangian. With H being a closed subgroup we obviously have a similar relation for t:

$$\mathrm{Ad}_h(t_{\mathrm{i}}) = h t_{\mathrm{i}} h^{-1} = \sum_{\mathrm{j}} \mathcal{E}_{\mathrm{ij}}(h) t_{\mathrm{j}} \tag{2.1.26}$$

One can now use (2.1.13) in order to rewrite the terms in (2.1.21) as linear combinations of x and t where one should note that the second term on the right-hand-side is a linear combination of t's only. The linear independence of the x's and t's tells us that their respective coefficients need to be equal. When writing the result in terms of the contractions

$$\begin{aligned} \widetilde{\mathrm{D}}_{\mathrm{a}\mu} &= \sum_{\mathrm{b}} \mathrm{D}_{\mathrm{ab}} \partial_\mu \pi_{\mathrm{b}} \quad , \\ \widetilde{\mathrm{E}}_{\mathrm{i}\mu} &= \sum_{\mathrm{b}} \mathrm{E}_{\mathrm{ib}} \partial_\mu \pi_{\mathrm{b}} \quad , \end{aligned} \tag{2.1.27}$$

rather simple expressions are obtained:

$$\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}'_{\mathrm{a}\mu} = h(\pi, g) \left(\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu} \right) h^{-1}(\pi, g) \tag{2.1.28}$$

$$\sum_{\mathrm{i}} t_{\mathrm{i}} \widetilde{\mathrm{E}}'_{\mathrm{i}\mu} = h(\pi, g) \left(\sum_{\mathrm{i}} t_{\mathrm{i}} \widetilde{\mathrm{E}}_{\mathrm{i}\mu} \right) h^{-1}(\pi, g) + \mathrm{i}\,[\partial_\mu h(\pi, g)] h^{-1}(\pi, g) \tag{2.1.29}$$

Using (2.1.22), we can see from (2.1.28) by comparing matching coefficients that

$$\mathrm{D}'_{\mathrm{ab}} = \sum_{\mathrm{c}} \mathcal{D}_{\mathrm{ac}}(h(\pi,g))\mathrm{D}_{\mathrm{cb}} \quad . \tag{2.1.30}$$

While the second equation could be evaluated similarly it is more useful to keep it in its present form as we shall see later. The contraction $\widetilde{\mathrm{D}}_{\mathrm{a}\mu} = \sum_{\mathrm{b}} D_{\mathrm{ab}}\partial_\mu \pi_{\mathrm{b}}$ is called the covariant derivative of the Goldstone fields and it transforms under a general $g \in G$ transformation through the linear representation $\mathcal{D}$ of the unbroken subgroup H, where the $h(\pi, g)$ defined in (2.1.19) is the group element of H corresponding to g. We already have seen that the fields $\tilde{\psi}$ themselves also transform linearly under general G transformations, namely just by the original linear action of $h(\pi, g)$ itself. However, $\partial_\mu \tilde{\psi}$ does not:

$$\partial_\mu \tilde{\psi}' = h(\pi,g)\partial_\mu \tilde{\psi} + (\partial_\mu h(\pi,g))\tilde{\psi} \tag{2.1.31}$$

The second term spoils the linearity of the transformation but it is essentially the same inhomogeneous term that appeared in (2.1.29). In fact we can cancel out the h^{-1} in (2.1.29) and replace it by the factor $\tilde{\psi}$ if we consider the term $\mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}\tilde{\psi}$ which transforms as

$$\begin{aligned} \mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}\tilde{\psi} \to \mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}'_{\mathrm{i}\mu}\tilde{\psi}' &= h(\pi,g)\mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}h^{-1}(\pi,g)h(\pi,g)\tilde{\psi} \\ &\quad - [\partial_\mu h(\pi,g)]h^{-1}(\pi,g)h(\pi,g)\tilde{\psi} \\ &= h(\pi,g)\left[\mathrm{i}\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}\tilde{\psi}\right] - [\partial_\mu h(\pi,g)]\tilde{\psi} \quad . \end{aligned} \tag{2.1.32}$$

The second term just cancels the problematic term from before if we add (2.1.31) and (2.1.32):

$$\begin{aligned} & \mathcal{D}_\mu\tilde{\psi} := \partial_\mu\tilde{\psi} + i\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}\tilde{\psi} \\ \to \quad & \left[\mathcal{D}_\mu\tilde{\psi}\right]' = h(\pi,g)\mathcal{D}_\mu\tilde{\psi} \end{aligned} \tag{2.1.33}$$

This is a covariant derivative of the fields $\tilde{\psi}$ and transforms just like the $\tilde{\psi}$ linearly by application of $h(\pi, g)$. Of course one can therefore apply the same derivative to $\mathcal{D}_\mu\tilde{\psi}$ itself and obtain a second order covariant derivative,

$$\begin{aligned} & \mathcal{D}_\nu\mathcal{D}_\mu\tilde{\psi} = \left[\partial_\nu + i\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\nu}\right]\left[\partial_\mu + i\sum_{\mathrm{i}} t_{\mathrm{i}}\widetilde{\mathrm{E}}_{\mathrm{i}\mu}\right]\tilde{\psi} \quad , \\ \to \quad & \left[\mathcal{D}_\nu\mathcal{D}_\mu\tilde{\psi}\right]' = h(\pi,g)\left[\mathcal{D}_\nu\mathcal{D}_\mu\tilde{\psi}\right] \end{aligned} \tag{2.1.34}$$

and so on. Similarly one can define a covariant derivative of $\widetilde{D}_{\mathrm{a}\mu}$. To that extend

one first calculates how the usual derivative transforms:

$$\begin{aligned}
\left[\sum_{\mathrm{a}} x_{\mathrm{a}} \partial_\nu \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right]' =& \partial_\nu \left[\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right]' \\
=& \partial_\nu \left[h(\pi,g) \sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu} h^{-1}(\pi,g)\right] \\
=& h(\pi,g) \sum_{\mathrm{a}} x_{\mathrm{a}} \left[\partial_\nu \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right] h^{-1}(\pi,g) \\
&+ [\partial_\nu h(\pi,g)] \left[\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right] h^{-1}(\pi,g) + h(\pi,g) \left[\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right] [\partial_\nu h^{-1}(\pi,g)] \\
=& h(\pi,g) \sum_{\mathrm{a}} x_{\mathrm{a}} \left[\partial_\nu \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right] h^{-1}(\pi,g) \\
&+ [\partial_\nu h(\pi,g)] h^{-1}(\pi,g) \left[h(\pi,g) \left(\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right) h^{-1}(\pi,g)\right] \\
&+ \left[h(\pi,g) \left(\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right) h^{-1}(\pi,g)\right] h(\pi,g) [\partial_\nu h^{-1}(pi,g)] \\
=& h(\pi,g) \sum_{\mathrm{a}} x_{\mathrm{a}} \left[\partial_\nu \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right] h^{-1}(\pi,g) \\
&+ \left[(\partial_\nu h(\pi,g)) h^{-1}(\pi,g)\,,\, h(\pi,g) \left(\sum_{\mathrm{a}} x_{\mathrm{a}} \widetilde{\mathrm{D}}_{\mathrm{a}\mu}\right) h^{-1}(\pi,g)\right]
\end{aligned} \tag{2.1.35}$$

For the final step we have used

$$h(\partial_\nu h^{-1}) = -(\partial_\nu h) h^{-1} \quad . \tag{2.1.36}$$

The first term can be simplified using (2.1.22) and yields a linear transformation through the adjoint representation of H on the x's. The second term however poses the same issue as in the case of the derivatives of $\tilde{\psi}$. The solution lies again in using that the transformation of $\mathrm{E}_{\mathrm{i}\mu}$ generates just the right nonlinear term that appears in this second term.

$$\begin{aligned}
\left[\mathrm{i} \sum_{\mathrm{i}} t_{\mathrm{i}} \widetilde{\mathrm{E}}_{\mathrm{i}\nu}\,,\, \sum_{\mathrm{b}} x_{\mathrm{b}} \widetilde{\mathrm{D}}_{\mathrm{b}\mu}\right]' =& h(\pi,g) \left[\mathrm{i} \sum_{\mathrm{i}} t_{\mathrm{i}} \widetilde{\mathrm{E}}_{\mathrm{i}\nu}\,,\, \sum_{\mathrm{b}} x_{\mathrm{b}} \widetilde{\mathrm{D}}_{\mathrm{b}\mu}\right] h^{-1}(\pi,g) \\
&- \left[(\partial_\nu h(\pi,g)) h^{-1}(\pi,g)\,,\, h(\pi,g) \left(\sum_{\mathrm{b}} x_{\mathrm{b}} \widetilde{\mathrm{D}}_{\mathrm{b}\mu}\right) h^{-1}(\pi,g)\right]
\end{aligned} \tag{2.1.37}$$

Once again the second terms cancel out if we add (2.1.35) and (2.1.37). Using

$\left[i \sum_i t_i \widetilde{E}_{i\nu} \, , \, \sum_b x_b \widetilde{D}_{b\mu} \right] = - \sum_{i,a,b} C_{iba} x_a \widetilde{E}_{i\nu} \widetilde{D}_{b\mu}$, one finds:

$$\begin{aligned} \left[\sum_a x_a \left(\partial_\nu \widetilde{D}_{a\mu} - \sum_{i,b} C_{iba} \widetilde{E}_{i\nu} \widetilde{D}_{b\mu} \right) \right]' &= \left[\sum_a h(\pi, g) x_a h^{-1}(\pi, g) \left(\partial_\nu \widetilde{D}_{a\mu} - \sum_{i,b} C_{iba} \widetilde{E}_{i\nu} \widetilde{D}_{b\mu} \right) \right] \\ &= \left[\sum_{a,c} x_c \mathcal{D}_{ca}(h(\pi, g)) \left(\partial_\nu \widetilde{D}_{a\mu} - \sum_{i,b} C_{iba} \widetilde{E}_{i\nu} \widetilde{D}_{b\mu} \right) \right] . \end{aligned} \tag{2.1.38}$$

We are led to define the covariant derivative of D:

$$\mathcal{D}_\nu \widetilde{D}_{a\mu} = \partial_\nu \widetilde{D}_{a\mu} - \sum_{i,b} C_{iba} \widetilde{E}_{i\nu} \widetilde{D}_{b\mu} \tag{2.1.39}$$

By comparing coefficients of x in (2.1.38) we see that this transforms just like D itself:

$$\left[\mathcal{D}_\nu \widetilde{D}_{a\mu} \right]' = \sum_b \mathcal{D}_{ab}(h(\pi, g)) \mathcal{D}_\nu \widetilde{D}_{b\mu} \tag{2.1.40}$$

More and more higher order expressions can be constructed in this manner and since it has been shown that any Lagrangian describing a system in this particular symmetry broken phase can be expressed in terms of these building blocks, we are left with the task of combining them into H-invariant terms. Of course we also need to respect the spacetime symmetries which affect the possible combinations in which derivatives appear.

2.2 Galilean Symmetry

In a non-relativistic setting the spacetime symmetries we need to incorporate are general translation invariance, meaning there is no explicit dependence of L on the spacetime coordinates, as well SO(3) invariance with respect to the spacial coordinates. Therefore a general Lagrangian build out of the covariant components that were constructed in the previous chapter has the following structure:

$$\begin{aligned}\mathcal{L} =& \sum_{a,b} v_a^{(1)} D_{ab} \dot{\pi}_b + \frac{1}{2} \sum_{a,b,c,d} g^t_{ab} D_{ac} D_{bd} \dot{\pi}_c \dot{\pi}_d + \sum_{ab} v_a^{(2)} \mathcal{D}_t \left(D_{ab} \dot{\pi}_b \right) \\ & - \frac{1}{2} \sum_{a,b,c,d} g^{\mathbf{x}}_{ab} D_{ac} D_{bd} \nabla \pi_c \cdot \nabla \pi_d - \sum_{a,b} \sum_{i=1}^{3} v_a^{(3)} \mathcal{D}_{x_i} \left(D_{ab} \partial_{x_i} \pi_b \right) \\ & - \frac{1}{2} \sum_{m,n} M_{mn} \tilde{\psi}_m \tilde{\psi}_n + \sum_{a,b,n} N_{an} D_{ab} \dot{\pi}_b \tilde{\psi}_n + \sum_n w_n \mathcal{D}_t \tilde{\psi}_n \\ & + \sum_{m,n} \left(T^{t,1}_{mn} \tilde{\psi}_m \mathcal{D}_t \tilde{\psi}_n + T^{t,2}_{mn} \mathcal{D}_t \tilde{\psi}_m \mathcal{D}_t \tilde{\psi}_n \right) + \sum_{m,n} G_{mn} \mathcal{D}_{\mathbf{x}} \tilde{\psi}_m \cdot \mathcal{D}_{\mathbf{x}} \tilde{\psi}_n + \dots \end{aligned} \tag{2.2.1}$$

The dots represent higher order terms in D and $\tilde{\psi}$. In order that $\mathcal{L}$ be G-symmetric we need to impose conditions on the coefficients in (2.2.1). By construction we only need to consider transformations under H and so the condition on v reads

$$\begin{aligned} & \forall\, h \in H, b = 1, \dots, (\dim(G) - \dim(H)) : \quad \sum_a v_a D'_{ab} = \sum_{a,c} v_a \mathcal{D}_{ac}(h) D_{cb} \overset{!}{=} \sum_a v_a D_{ab} \\ \Leftrightarrow & \forall\, h \in H, b = 1, \dots, (\dim(G) - \dim(H)) : \quad \sum_a v_a \mathcal{D}_{ab}(h) = v_b \\ \Leftrightarrow & \forall\, b = 1, \dots, (\dim(G) - \dim(H)), i = 1, \dots, \dim(H) : \quad \sum_a v_a C_{iab} = 0 \end{aligned} \tag{2.2.2}$$

The last form of this condition follows by differentiating the second line, using (2.1.25) and $\mathcal{D}_{ab}(e_H) = \delta_{ab}$, where e_H is the identity element in H. Similarly we have

$$\begin{aligned} & \forall\, b,c = 1, \dots, (\dim(G) - \dim(H)), i = 1, \dots, \dim(H) : \\ & \qquad \sum_a (C_{iab}\, g^{t/\mathbf{x}}_{ac} + C_{iac}\, g^{t/\mathbf{x}}_{ba}) = 0 \end{aligned} \tag{2.2.3}$$

The mass matrix M has to be invariant under the same linear representation of H under which the fundamental fields transform:

$$\begin{aligned} & \sum_{m,n} M_{mn} \tilde{\psi}'_m \tilde{\psi}'_n = \sum_{m,n,p,q} M_{mn} [h]_{mp} [h]_{nq} \tilde{\psi}_p \tilde{\psi}_q \overset{!}{=} \sum_{p,q} M_{pq} \tilde{\psi}_p \tilde{\psi}_q \\ \Leftrightarrow \quad & \forall\, i = 1, \dots, \dim(H),\ p,q = 1, \dots, N : \sum_m (M_{pm} [t_i]_{mq} + M_{mq} [t_i]_{mp}) = 0 \end{aligned} \tag{2.2.4}$$

Again, the second line is obtained by differentiating the first. The same condition applies to T and G as well of course. Finally, the condition for N reads

$$\begin{aligned}
&\sum_{a,b,n} N_{an} D'_{ab} \dot{\pi}_b \tilde{\psi}'_n = \sum_{a,b,c,m,n} N_{an} \mathcal{D}_{ac}(h) D_{cb} \dot{\pi}_b [h]_{nm} \tilde{\psi}'_m \stackrel{!}{=} \sum_{c,b,m} N_{cm} D_{cb} \dot{\pi}_b \tilde{\psi}_m \\
&\Leftrightarrow \forall\, b = 1, \ldots, (\dim(G) - \dim(H)),\ m = 1, \ldots, N: \quad \sum_{n,a} (i N_{bn} [t_i]_{nm} + N_{am} C_{iab}) = 0
\end{aligned} \tag{2.2.5}$$

At this point we should address the fact, that a general Lagrangian constructed in this way will include terms that depend on the values of the Goldstone-fields themselves and not just their derivatives. This contradicts the argument we gave earlier for the absence of any such terms, which shows that this construction is more general than one which starts with fundamental fields and explicitly reparameterizes them in terms of Goldstone-fields and massive fields. The latter will in general yield both gapless and gapped degrees of freedom in the quantized Hamiltonian. If one is only interested in the low energy limit then the gapped modes could be integrated out to give an effective theory for the gapless modes. Alternatively, one can construct a low energy effective Lagrangian in the first place by keeping only the Goldstone fields in the general construction outlined above. The kind of allowed terms go beyond what we would obtain by explicit reparameterization and take into account all of the possible effective interactions one could generate by integrating out massive degrees of freedom. This general construction is therefore really a phenomenological approach to the dynamics of a system in the vicinity of a given phase. It correctly determines the number of relevant modes, their dispersion relations in terms of powers of (quasi-)momentum and the structure of their interactions. It is also very suitable for an expansion of $\mathcal{L}$ in powers of fields since the building blocks are already given in terms of power series.

3 Constrained Quantization [1]

3.1 Constructing the Hamiltonian

The usual scheme for passing from the Lagrangian formalism to the Hamiltonian formalism via a Legendre transformation relies on the Lagrangian being a convex function of the time derivatives of the canonical variables. If this condition is violated then the transition is not as simple, yet still possible. In the case of relativistic theories, Lorentz invariance forces the Lagrangian to be quadratic and therefore convex in the time derivatives. In the non-relativistic case we will have these time derivatives appearing in first order only and therefore have to deal with non-convex Lagrangians. This means that the Legendre transformation can still, in some sense, be naively performed but is neither well defined nor invertible and the usual deduction of the Hamiltonian equations of motion from the Lagrangian equations of motion is no longer valid. In such cases some of the Euler-Lagrange equations and the defining equations for conjugate momenta will actually put constraints on the naively defined phase space instead of merely providing equations of motion. The physical phase space will then be the symplectic submanifold of the naive phase space which is defined by these constraints and the naive Hamiltonian will coincide with the physical Hamiltonian $\mathcal{H}$ on this submanifold. Since the dynamics in the Hamiltonian formalism is generated by derivatives of $\mathcal{H}$ in the form of poisson brackets, it is not sufficient to know the Hamiltonian as a function on the submanifold. Additionally, one also needs to know the normal component of its gradient to this hypersurface.

A basic example that illustrates these points is the Lagrangian for a classical particle with mass m and charge q moving in the x-y plane, subject to an external magnetic field and an arbitrary scalar potential V in the limit where the fields strength B is large and the kinetic term can be dropped. In this approximation the Lagrangian L reads

$$L = \frac{qB}{2c}(x\dot{y} - y\dot{x}) - V(x,y) \quad . \tag{3.1.1}$$

This gives two legitimate first order equations of motion:

$$\dot{x} = -\frac{c}{qB}\partial_y V(x,y) \quad , \quad \dot{y} = \frac{c}{qB}\partial_x V(x,y) \tag{3.1.2}$$

However, the standard definition of the canonical momenta does not provide an

[1] For more in-depth discussions see for example [6, 9, 10, 15]

invertible relation between the velocities and these momenta:

$$p_x = \frac{\partial L}{\partial \dot{x}} = -\frac{c}{qB} y \quad , \quad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{c}{qB} x \tag{3.1.3}$$

While the phase space naively consists of the 2 pairs of conjugate variables (x, p_x) and (y, p_y), it is clear that the structure of the Lagrangian itself already restricts the actual number of degrees of freedom to 2 variables, since the momenta are just functions of position. We can nevertheless proceed and define the naive Hamiltonian in the standard fashion:

$$H(Q, P) = \sum_{\mathrm{i}} P_{\mathrm{i}} \dot{Q}_{\mathrm{i}} - L(Q, \dot{Q}) \tag{3.1.4}$$

In this case on obtains

$$H(x, y, p_y, p_y) = V(x, y) \quad . \tag{3.1.5}$$

It must be noted that we have used the constrains, that is eq. (3.1.3), in order to cancel out the terms involving velocities. First of all this demonstrates the general fact that the naive Hamiltonian can alway be written in terms of Q and P only. This is because either there is an invertible relation between P and $\dot{Q}$ or, if there is not, the terms containing $\dot{Q}$ will cancel out when using the constraints. The second important observation is that this naive Hamiltonian coincides with the physical Hamiltonian, that is the one which reproduces the correct dynamical equations, only on the submanifold defined by the constraints in phase space, since by using the constraints we effectively moved onto this physical phase space ($\mathcal{PS}$). Using the constraints allows for an interchange between momenta and coordinates which alters the derivatives of H with respect to these variables. Therefore one cannot simply assume the usual Hamiltonian equations of motion $\dot{Q} = \partial H / \partial P$ and $\dot{P} = -\partial H / \partial Q$. If we write every such constraint in the form $\varphi_i(Q, P) = 0$, we can use the fact that interchanging terms in the Hamiltonian by using the constraint conditions just amount to adding terms proportional to $\varphi_i(Q, P)$. This goes the other way around as well. Any two H and H' that differ only by a linear combination of these φ_i will be equal on the physical phase space where $\varphi_i = 0$ but differ anywhere else on the naive phase space. The physical Hamiltonian H' which provides the correct dynamics by the standard Hamiltonian equations of motion should therefore take the form

$$H'(Q, P) = H(Q, P) + \sum_{\mathrm{i}} \lambda_{\mathrm{i}}(Q, P) \varphi_{\mathrm{i}}(Q, P) \quad , \tag{3.1.6}$$

where the λ_{i} are suitable Lagrange multipliers that need to be determined. The Hamiltonian equations of motion are of course expressable in terms of Poisson

brackets. For any function $f(Q,P)$ we have

$$\dot{f} = \{f, H'\} \quad , \quad \{f,g\} := \sum_{\mathrm{i}} \left(\frac{\partial f}{\partial Q_{\mathrm{i}}} \frac{\partial g}{\partial P_{\mathrm{i}}} - \frac{\partial f}{\partial P_{\mathrm{i}}} \frac{\partial g}{\partial Q_{\mathrm{i}}} \right) \tag{3.1.7}$$

In order that the dynamics be consistent with the constraints we need to impose the conditions

$$\begin{aligned} & \{\varphi_{\mathrm{j}}, H'\} = \{\varphi_{\mathrm{j}}, H\} + \sum_{\mathrm{i}} \lambda_{\mathrm{i}} \{\varphi_{\mathrm{j}}, \varphi_{\mathrm{i}}\} + \{\varphi_{\mathrm{j}}, \lambda_{\mathrm{i}}\} \varphi_{\mathrm{i}} \stackrel{!}{=} 0 \quad \forall (Q,P) \in \mathcal{PS} \\ \Leftrightarrow \quad & \{\varphi_{\mathrm{j}}, H'\} = \{\varphi_{\mathrm{j}}, H\} + \sum_{\mathrm{i}} \lambda_{\mathrm{i}} \{\varphi_{\mathrm{j}}, \varphi_{\mathrm{i}}\} \stackrel{!}{=} 0 \qquad \forall (Q,P) \in \mathcal{PS} \end{aligned} \tag{3.1.8}$$

Ideally, these would just fix the Lagrange multipliers but in general there are several possibilities at this point. If the matrix $M_{\mathrm{ij}} = \{\varphi_{\mathrm{i}}, \varphi_{\mathrm{j}}\}$ is invertible on $\mathcal{PS}$ then we refer to the φ as a set of second class constraints and (3.1.8) can be explicitly solved for the λ_{i}. If M_{ij} is singular then one can mix the set of constraints in an invertible manner by defining $\varphi'_{\mathrm{i}} = \sum_{\mathrm{j}} T_{\mathrm{ij}} \varphi_{\mathrm{j}}$ with T_{ij} invertible such that the set φ' splits into two subsets $\varphi'_{2^{\mathrm{nd}}\mathrm{class}}$ and $\varphi'_{1^{\mathrm{st}}\mathrm{class}}$. These have the properties

$$\begin{aligned} & M'_{\mathrm{ij}} = \{ (\varphi'_{2^{\mathrm{nd}}\mathrm{class}})_{\mathrm{i}} , (\varphi'_{2^{\mathrm{nd}}\mathrm{class}})_{\mathrm{j}} \} \qquad \text{invertible,} \\ & \{ (\varphi'_{1^{\mathrm{st}}\mathrm{class}})_{\mathrm{i}} , \varphi'_{\mathrm{j}} \} = 0 \quad . \end{aligned} \tag{3.1.9}$$

That is, the first class constraints commute with any other constraint. The consistency conditions for the second class constraints determine their respective Lagrange multipliers appearing in H'. Since M_{ij} is always skew symmetric, there have to be an even number of second class constraints because otherwise this matrix could not be invertible. The consistency conditions for the first class constraints read

$$\{ (\varphi'_{1^{\mathrm{st}}\mathrm{class}})_{\mathrm{i}} , H \} \stackrel{!}{=} 0 \qquad \forall (Q,P) \in \mathcal{PS} \tag{3.1.10}$$

If any of these is not trivially satisfied then it has to be added as another, so called secondary constraint to the original set of primary constraints. This process will then be repeated, adding more and more constraints until, after a finite number of repetitions, the consistency conditions for the first class constraints are either contradictory or trivially satisfied. The first case implies an inconsistent dynamical system with the inconsistency already present at the level of the Euler-Lagrange equations, an example being the Lagrangian $L(Q,\dot{Q}) = Q$. In the second case each of the first class constraints commutes with H' and is therefore a generator of a symmetry of H'. In the field-theoretic setting we will see that the constraints are local functions and therefore generate gauge symmetries of the Hamiltonian. This explains why the Lagrange multipliers of first class constraints remain undetermined, since adding these constraints to H is equivalent to evolving the system

with a Hamiltonian that does not include them and then subsequently performing a finite gauge transformation. Since different values of a variable that are related by a gauge transformation need to be identified in order to obtain the physical degrees of freedom, the two different Hamiltonians become in fact equivalent after modding out these gauge symmetries.

Equations of motion for any function $f(Q,P)$ are now obtained by calculating $\{f,H'\}$. Constraints can be used freely in order to interchange variables after the Poisson brackets have been evaluated but not before. There is however an alternative to using the full Hamiltonian H', which also allows for the constraints to be used freely at any stage without affecting the equations of motion. This method is known as Dirac brackets and will be explored in the following section.

3.2 Dirac Brackets

Since we can choose the Lagrange multipliers of any first class constraints in H' to vanish and by denoting the second class constraints from now on merely as φ we are left with

$$\begin{aligned} H' &= H + \sum_{\mathrm{i}} \lambda_{\mathrm{i}} \varphi_{\mathrm{i}} \\ &= H - \sum_{\mathrm{i,j}} M^{-1}_{\mathrm{ij}} \{\varphi_{\mathrm{j}}, H\} \varphi_{\mathrm{i}} \quad . \end{aligned} \tag{3.2.1}$$

In the second line the Lagrange multipliers of the second class constraints were explicitly expressed by solving the corresponding consistency conditions (3.1.8). Now we have

$$\dot{f} = \{f,H'\} = \{f,H\} - \sum_{\mathrm{i,j}} M^{-1}_{\mathrm{ij}} \{\varphi_{\mathrm{j}}, H\} \{f, \varphi_{\mathrm{i}}\} \tag{3.2.2}$$

This suggests that we could use the naive Hamiltonian H instead of H' if we replace all Poisson brackets by Dirac brackets, which are defined as

$$\{f,g\}_{\mathrm{DB}} \coloneqq \{f,g\} - \sum_{\mathrm{i,j}} \{f,\varphi_{\mathrm{i}}\} M^{-1}_{\mathrm{ij}} \{\varphi_{\mathrm{j}}, g\} \quad . \tag{3.2.3}$$

Since the Dirac bracket of any constraint with any other function clearly vanishes, one can use constraints before calculating Dirac brackets without affecting the result when it is evaluated on $\mathcal{PS}$.

This approach is particularly relevant if we want to quantize the classical theory because the canonical procedure dictates that the canonical variables should be replaced by operators and Poisson brackets by commutators. This poses an issue since Poisson brackets of constraints with other functions generally do not vanish even after applying all of the constraints. Therefore the corresponding commutators would yield non-vanishing operators on the physical Hilbert space while the constraints themselves have to vanish on any physical state. This is contradictory

and calls for a modification of the quantization procedure in such circumstances. By replacing the Dirac brackets instead of Poisson brackets with commutators this issue is completely solved. The downside however is that the canonical commutations relations of the Q's and P's are now altered and in general not that easy to work with. As it turns out this can as well be circumvented by choosing canonical variables that are better suited to the situation. As Maskawa and Nakajima showed in [11], one can always perform a canonical transformation on the original set of canonical variables in a way such that the new set is split into two subsets (Q,P) and $(\mathcal{Q}, \mathcal{P})$ of canonically conjugate pairs. The crucial property of these subsets is that the even number of second class constraints in these coordinates just read $\mathcal{Q} = 0$ and $\mathcal{P} = 0$. Therefore in these coordinates we have $M = S^{(n)}$, where $S^{(n)}$ is the canonical symplectic metric in n dimensions and n is the number of second class constraints. This implies $M^{-1} = -M$ and when we calculate Dirac brackets in these coordinates we find a simple relation:

$$\begin{aligned}\{f,g\}_{\mathrm{DB}} &= \{f,g\} + \sum_{\mathrm{i}} \left[\{f,\mathcal{Q}_{\mathrm{i}}\}\{\mathcal{P}_{\mathrm{j}},g\} - \{f,\mathcal{P}_{\mathrm{i}}\}\{\mathcal{Q}_{\mathrm{j}},g\}\right] \\ &= \{f,g\} - \sum_{\mathrm{i}} \left[\frac{\partial f}{\partial \mathcal{Q}_{\mathrm{i}}}\frac{\partial g}{\partial \mathcal{P}_{\mathrm{i}}} - \frac{\partial f}{\partial \mathcal{P}_{\mathrm{i}}}\frac{\partial f}{\partial \mathcal{Q}_{\mathrm{i}}}\right] \\ &= \sum_{\mathrm{i}} \left[\frac{\partial f}{\partial \mathrm{Q}_{\mathrm{i}}}\frac{\partial g}{\partial \mathrm{P}_{\mathrm{i}}} - \frac{\partial f}{\partial \mathrm{P}_{\mathrm{i}}}\frac{\partial f}{\partial \mathrm{Q}_{\mathrm{i}}}\right]\end{aligned} \tag{3.2.4}$$

This is just the usual Poisson bracket in terms of the unconstrained variables (P,Q) only. Since by construction Dirac brackets commute with the application of constraints, we can just set $\mathcal{Q} = 0$ and $\mathcal{P} = 0$ in the naive Hamiltonian H after rewriting it in terms of these new variables and then calculate the equations of motions of the remaining variables in terms of standard Poisson brackets. These standard Poisson brackets are to be replaced by commutators and therefore the canonical commutation relations are preserved. Of course one could just as well perform a canonical transformation on the complete set of operators that satisfy some altered commutation relations in order to recover the canonical ones while simultaneously having a subset of the new operators representing the constraints.

3.3 Field Theory

In a field theoretical setting the formalism of constrained quantization essentially remains the same as for point particles which are after all just the 0+1 dimensional case of a field theory. The first difference is that the Lagrangian will now be written as an integral over a local Lagrangian density

$$L = \int \mathrm{d}^3x \mathcal{L}(\psi(\mathbf{x}), \dot{\psi}(\mathbf{x}), \nabla\psi(\mathbf{x})) \quad . \tag{3.3.1}$$

The canonical momenta P are analogously defined as

$$P_{\mathrm{i}}(\mathbf{x}) := \partial\mathcal{L}(\mathbf{x}/\partial\dot{\psi}_{\mathrm{i}}(\mathbf{x}) \quad . \tag{3.3.2}$$

For general functions A of the field variables which take the form

$$A = \int \mathrm{d}^3x \mathcal{A}(\psi(\mathbf{x}), P(\mathbf{x}), \nabla\psi(\mathbf{x}), \nabla P(\mathbf{x})) \tag{3.3.3}$$

one defines the Poisson bracket as follows:

$$\{A, B\} \coloneqq \int \mathrm{d}^3x \sum_{\mathrm{i}} \left[\frac{\delta\mathcal{A}}{\delta\psi_{\mathrm{i}}}\frac{\delta\mathcal{B}}{\delta P_{\mathrm{i}}} - \frac{\delta\mathcal{B}}{\delta\psi_{\mathrm{i}}}\frac{\delta\mathcal{A}}{\delta P_{\mathrm{i}}}\right](\mathbf{x}) \tag{3.3.4}$$

The variational derivative $\delta\mathcal{F}/\delta f$ is defined as

$$\frac{\delta\mathcal{F}}{\delta f}(\mathbf{x}) = \frac{\partial\mathcal{F}}{\partial f(\mathbf{x})} - \sum_{\mathrm{i}} \nabla_{\mathrm{i}} \frac{\partial\mathcal{F}}{\partial(\nabla_{\mathrm{i}} f(\mathbf{x}))} \tag{3.3.5}$$

With these definitions everything works analogously to the procedure outlined in the previous section. The naive Hamiltonian will be

$$H = \int \mathrm{d}^3x \sum_{\mathrm{i}} P_{\mathrm{i}}(\mathbf{x})\dot{\psi}_{\mathrm{i}}(\mathbf{x}) - \mathcal{L}(\psi(\mathbf{x}), \dot{\psi}(\mathbf{x}), \nabla\psi(\mathbf{x})) \quad , \tag{3.3.6}$$

with the $\dot{\psi}$ either being expressed in terms of fields and momenta or canceling out in cases where constraints $\varphi_{\mathrm{i}}(\psi(\mathbf{x}), P(\mathbf{x}))$ are present. The physical Hamiltonian can then be written as

$$H' = H + \int \mathrm{d}^3x \sum_{\mathrm{i}} \lambda_{\mathrm{i}}(\psi(\mathbf{x}), P(\mathbf{x}))\, \varphi_{\mathrm{i}}(\psi(\mathbf{x}), P(\mathbf{x})) \tag{3.3.7}$$

and the consistency conditions may lead to new constraints until we are left with as consistent set of first class constraints and a set of second class constraints. The spacial argument in the constraints can be viewed as part of the index and can therefore think of every φ_{i} as a (continuous) set of constraints. Any $(\varphi_{1^{\mathrm{st}}\mathrm{class}})_{\mathrm{i}}(\mathbf{x})$ generates a symmetry of H via the Poisson bracket

$$\delta H = \epsilon\{(\varphi_{1^{\mathrm{st}}\mathrm{class}})_{\mathrm{i}}(\mathbf{x}), H\} = 0 \tag{3.3.8}$$

and since it generates a local transformation of the fields, we see that it indeed corresponds to a gauge symmetry.

The necessity for using this more general framework of quantization in the context of field theory can be traced back to the structure of the Lagrangian (2.2.1), which is in general of first order in time derivatives due to the symmetry group being the Galilean group. A standard example for a Lorentz symmetric system would

be any scalar field theory of the form

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \tag{3.3.9}$$

where V is a potential that only depends on ϕ and not its derivatives. Standard canonical quantization then yields the Hamiltonian

$$\mathcal{H}_{\text{scalar}} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi), \tag{3.3.10}$$

with π being the conjugate momentum of ϕ. The Feynman path integral constructed from this Hamiltonian then takes the form

$$Z = \int \mathcal{D}\phi \mathcal{D}\pi \exp\left[\mathrm{i}\int_x \pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]. \tag{3.3.11}$$

The integrand formally looks like a Legendre transform and since the Hamiltonian and the Lagrangian are just Legendre transforms of each other, we expect the action to be expressable in terms of the Lagrangian $\mathcal{L}_{\text{scalar}}$ Since the integrand is quadratic in π, we can make this explicit by completing the square and performing the integration over π which indeed yields

$$Z = \int \mathcal{D}\phi \exp\left[\mathrm{i}\int_x \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right]. \tag{3.3.12}$$

However, the integrand in the path integral formulation will in general not be quadratic in the momenta. This is the case for the non-relativistic theories that will be discussed in the following chapters in particular. We will therefore make the ansatz that the integrand can be taken as a classical Lagrangian without integrating out any fields and instead treating all of them as independent. This effectively doubles the number of degrees of freedom artificially in the sense that canonical quantization introduces a conjugate momentum for each of the fields, which are in reality of course already conjugate pairs. It is therefore clear that constraints must appear in order to reduce the number of degrees of freedom. We will show that the more general constrained quantization reproduces the correct Hamiltonian and then use the more general freedom of field re-definitions in the Lagrangian framework in order to apply the coset construction of the previous chapter.

4 Spin-1 BEC with Quadratic Zeeman Effect

In this chapter we will apply the techniques that were presented in the previous two chapters to a gas of spin-1 particles at very low temperature. This system will be modeled by a Hamiltonian describing non-relativistic particles with a free dispersion relation that is quadratic in momenta, interacting via a two-particle contact interaction which is spin-dependent and exhibiting a quadratic Zeeman shift induced by an external magnetic field. Instead of directly writing down this Hamiltonian we will start with a suitable Lagrangian and show that the correct Hamiltonian can be derived from it. This will first of all demonstrate the procedure of constrained quantization in a comparatively simple example before we move on to more complicated cases. It also allows us to do the reparameterization of the Lagrangian in terms of Goldstone-fields and massive fields explicitly instead of just looking at the general construction of low energy effective Lagrangians based on the symmetry breaking. The latter could also be done by just looking at the symmetries and local ground states of the Hamiltonian. First we will look at a model that includes a density-density interaction but does not take into account the spin structure of the interaction. After that we will include the spin structure and compare the results of both the explicit calculation in terms of reparameterized fields, as well as the results of the general coset construction in each of these cases. The symmetry groups of the two Lagrangians differ, which affects the structure of the allowed terms in the coset construction.

4.1 U(N)-Symmetric Lagrangian

We postulate that the correct Lagrangian density for spin-N particles with density interaction written in terms of N complex fields $(\psi_1, \ldots, \psi_N) \in \mathbb{C}^N$ takes the form

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} \left[\psi_i^* \left(i\partial_t + \frac{\Delta}{2M} \right) \psi_i + \psi_i \left(-i\partial_t + \frac{\Delta}{2M} \right) \psi_i^* \right] - \frac{g}{2} \left[\sum_{i=1}^{N} \psi_i^* \psi_i \right]^2 + \mu \sum_{i=1}^{N} \psi_i^* \psi_i \tag{4.1.1}$$

The last term is the total particle number density operator times a chemical potential which acts as a Lagrange multiplier that fixes the total number density to a given value n and whose value is to be determined together with the ground state field expectation value when solving the classical equations of motion which correspond to a stationary phase analysis in the path integral formalism. It is important to note that this term does not alter the symmetry of $\mathcal{L}$. In fact the symmetry group of this Lagrangian is U(N) and one possible choice of a basis for the Lie algebra is

the following set of nine complex, hermitian matrices:

$$x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix}, x_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$t_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, t_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}, t_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, t_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1.2}$$

The reason for using this particular basis will become clear once we identify the ground states of this system. $\mathcal{L}$ is of course a real function and can be written in terms of 2N real fields, namely the real and imaginary parts of the ψ_i. If we define

$$\psi_\mathrm{i} = (\psi_{\mathrm{i}1} + \mathrm{i}\psi_{\mathrm{i}2})/\sqrt{2}, \tag{4.1.3}$$

we obtain

$$\begin{aligned} \mathcal{L} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \left[\frac{1}{2}(\psi_{\mathrm{i}2}\partial_\mathrm{t}\psi_{\mathrm{i}1} - \psi_{\mathrm{i}1}\partial_\mathrm{t}\psi_{\mathrm{i}2}) - \frac{1}{4m}(\nabla\psi_{\mathrm{i}1}\cdot\nabla\psi_{\mathrm{i}1} + \nabla\psi_{\mathrm{i}2}\cdot\nabla\psi_{\mathrm{i}2}) \right] \\ - \frac{g}{8}\left(\sum_{\mathrm{i}=1}^{\mathrm{N}} (\psi_{\mathrm{i}1}^2 + \psi_{\mathrm{i}2}^2) \right)^2 + \frac{\mu}{2}\sum_{\mathrm{i}=1}^{\mathrm{N}} (\psi_{\mathrm{i}1}^2 + \psi_{\mathrm{i}2}^2). \end{aligned} \tag{4.1.4}$$

4.1.1 Hamiltonian in the Fundamental Field Formulation

In order to verify that $\mathcal{L}$ produces the correct Hamiltonian and to illustrate constrained quantization, we will now explicitly quantize the theory in terms of the fundamental fields. Because the action S is the spacetime integral of $\mathcal{L}$, the term in (4.1.4) involving time derivatives can be exchanged for $\sum_{\mathrm{i}=1}^{\mathrm{N}} \psi_{\mathrm{i}2}\partial_\mathrm{t}\psi_{\mathrm{i}1}$ by partial integration with respect to the time variable. The definition of the canonical momenta yields N constraints:

$$\varphi_{\mathrm{i},1} = \psi_{\mathrm{i},2} - P_{\mathrm{i},1} \quad , \quad \varphi_{\mathrm{i},2} = P_{\mathrm{i},2} \quad , \text{with} \quad \mathrm{i} = 1, \ldots, \mathrm{N} \quad . \tag{4.1.5}$$

By calculating their Poisson brackets we find that $M = S^{(2\mathrm{N})}$, i.e. the symplectic metric and since $\det(S) = 1$, we have N second class constraints. In this case it is actually easier to just calculate the Dirac brackets instead of changing variables since they turn out to be equal to the Poisson brackets. The naive Hamiltonian density reads

$$\mathcal{H} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{1}{4m}\left(\nabla\psi_{\mathrm{i}1}\cdot\nabla\psi_{\mathrm{i}1} + \nabla P_{\mathrm{i}1}\cdot\nabla P_{\mathrm{i}1}\right) + \frac{g}{8}\left(\sum_{\mathrm{i}=1}^{3} \left(\psi_{\mathrm{i}1}^2 + P_{\mathrm{i}1}^2\right) \right)^2 - \frac{\mu}{2}\sum_{\mathrm{i}=1}^{\mathrm{N}} \left(\psi_{\mathrm{i}1}^2 + P_{\mathrm{i}1}^2\right).$$

(4.1.6)

Here we have already used the constraints in order to replace half of the fields with momenta. The remaining fields and momenta obey canonical commutation relations if we replace Dirac brackets by (-i) times the commutators and so we are led to define the creation and annihilation operators $\Phi^\dagger$ and Φ implicitly via

$$\psi_{i1} = \frac{1}{\sqrt{2}}\left(\Phi_i^\dagger + \Phi_i\right) \qquad , \qquad P_{i1} = \frac{i}{\sqrt{2}}\left(\Phi_i^\dagger - \Phi_i\right) \quad . \tag{4.1.7}$$

These obey the standard commutation relations

$$\left[\Phi_i(\mathbf{x}), \Phi_j^\dagger(\mathbf{y})\right] = \delta_{i,j}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \quad . \tag{4.1.8}$$

Inserting these relations into (4.2.9) and using partial integration with respect to the spacial derivatives in the first term gives

$$\mathcal{H} = \sum_{i=1}^{N} \Phi_i^\dagger \left(-\frac{\nabla^2}{2m}\Phi_i\right) + \frac{g}{2}\left(\sum_{i=1}^{N}\Phi_i^\dagger\Phi_i\right)^2 - \mu\sum_{i=1}^{N}\Phi_i^\dagger\Phi_i \qquad , \tag{4.1.9}$$

which is indeed the correct Hamiltonian density including the chemical potential term. The coherent state path integral constructed from this Hamiltonian takes the form

$$\int \mathcal{D}(\bar{\Psi},\Psi)\exp\left[\int_x \sum_{i=1}^{3}\left[\frac{1}{2}i(\psi_i^*\partial_t\psi_i - \psi_i\partial_t\psi_i^*) - \frac{1}{2m}\nabla\psi_i^*\cdot\nabla\psi_i\right] - \frac{g}{2}\left(\sum_{i=1}^{3}\psi_i^*\psi_i\right)^2 + \mu\sum_{i=1}^{N}\psi_i^*\psi_i\right], \tag{4.1.10}$$

where the complex fields Ψ are just the eigenvalues obtained by applying annihilation operators Φ to coherent states. The replacement $\psi_i = (\psi_{i1} + i\psi_{i2})/\sqrt{2}$ is therefore just the transition from creation and annihilation operators to canonical fields and momenta as formulated in the path integral setting if we identify $\psi_{i2} = P_{i1}$. In this sense the action in (4.1.10) is still written in terms of the Hamiltonian and one could make the transition to a Lagrangian by integrating over the momenta only in order to obtain a function of canonical fields and their derivatives. However, the approach we have taken here is to view the integrand in (4.1.10) as a Lagrangian by treating the momenta P_{i1} as canonical fields ψ_{i2} instead at the cost of introducing constraints at the transition to the Hamiltonian formalism. These have to arise because we already know that there are really only three independent canonical fields ψ_{i1} and their conjugate momenta. The advantage of this point of view is that one can re-construct the Hamiltonian after changing from fundamental fields to Goldstone- and massive fields by following the same procedure of constraint quantization as before, since changing variables in the Lagrangian results in a Hamiltonian which could as well be obtain by a canonical transformation from

the original Hamiltonian and is therefore completely equivalent. This procedure is of course equivalent to re-defining field variables in the path integral in order to diagonalize the quadratic part of the integrand, but one does not have to deal with functional determinants and a standardized procedure breaks this transformation down into a step of transformations that can be determined more easily.

The connection with the path integral formalism allows us to compare and match the parameters of the Goldstone-Lagrangian obtained by explicit reparameterization with the coefficients that we need to introduce in the general construction and therefore to combine the advantages of both approaches.

4.1.2 Explicit Parameterization in the Symmetry Breaking Ground State

By employing a stationary phase analysis and taking into account that the expectation value of the total particle number density is a fixed number n, one finds that there is only one phase and that one representative ground state, prior to quantum corrections, is characterized by the set of order parameters

$$\langle \Psi^{c} \rangle_{U(N)} = \left(\sqrt{n}, 0, \ldots, 0\right) \tag{4.1.11}$$

with $\Psi^{c} = [\psi_1, \ldots, \psi_N]$ the set of complex fields. Simultaneously one finds that the chemical potential must take the value

$$\mu = gn \quad . \tag{4.1.12}$$

In the case N=3 we have $\langle \Psi^{c} \rangle_{U(3)} = \left(\sqrt{n}, 0, 0\right)$ and the Lie algebra basis (4.1.2) is the result of determining the linear subspace of $\mathfrak{u}(3)$ which leaves the set of order parameters invariant and then choosing the other basis elements such that the conditions (2.1.7)-(2.1.8) are satisfied. The set $\{t\}$ generates a representation of the unbroken subgroup H, which in this case is U(2). The set $\{x\}$ are the broken generators of U(3). In order to perform the change of field variables we have to switch to the purely real fields

$$\Psi^{r} = [\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}, \psi_{31}, \psi_{32}] \quad , \tag{4.1.13}$$

where we have

$$\langle \Psi^{r} \rangle_{U(3)} = \left(\sqrt{2n}, 0, 0, 0, 0, 0\right) \quad . \tag{4.1.14}$$

Also the set of generators must be replaced by their counterparts in the real repre-

sentation, which are purely imaginary due to our choice of convention, see 2.1:

$$
\begin{aligned}
x_1^{\mathrm{r}} &= \begin{bmatrix} 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0\\ 0&\mathrm{i}&0&0&0&0\\ -\mathrm{i}&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, x_2^{\mathrm{r}} = \begin{bmatrix} 0&0&-\mathrm{i}&0&0&0\\ 0&0&0&-\mathrm{i}&0&0\\ \mathrm{i}&0&0&0&0&0\\ 0&\mathrm{i}&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, x_3^{\mathrm{r}} = \begin{bmatrix} 0&0&0&0&0&\mathrm{i}\\ 0&0&0&0&-\mathrm{i}&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&\mathrm{i}&0&0&0&0\\ -\mathrm{i}&0&0&0&0&0 \end{bmatrix}\\
x_4^{\mathrm{r}} &= \begin{bmatrix} 0&0&0&0&-\mathrm{i}&0\\ 0&0&0&0&0&-\mathrm{i}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \mathrm{i}&0&0&0&0&0\\ 0&\mathrm{i}&0&0&0&0 \end{bmatrix}, x_5^{\mathrm{r}} = \begin{bmatrix} 0&\mathrm{i}&0&0&0&0\\ -\mathrm{i}&0&0&0&0&0\\ 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0\\ 0&0&0&0&0&\mathrm{i}\\ 0&0&0&0&-\mathrm{i}&0 \end{bmatrix}, t_6^{\mathrm{r}} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\mathrm{i}\\ 0&0&0&0&-\mathrm{i}&0\\ 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0 \end{bmatrix}\\
t_7^{\mathrm{r}} &= \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-\mathrm{i}&0&0\\ 0&0&0&0&0&-\mathrm{i}\\ 0&0&\mathrm{i}&0&0&0\\ 0&0&0&\mathrm{i}&0&0 \end{bmatrix}, t_8^{\mathrm{r}} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0\\ 0&0&0&0&0&-\mathrm{i}\\ 0&0&0&0&\mathrm{i}&0 \end{bmatrix}, t_9^{\mathrm{r}} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0\\ 0&0&0&0&0&\mathrm{i}\\ 0&0&0&0&-\mathrm{i}&0 \end{bmatrix},
\end{aligned}
\tag{4.1.15}
$$

We can now determine the massive field $\tilde{\boldsymbol{\psi}}$ via the condition (2.1.5):

$$
\begin{aligned}
&\forall \mathrm{a} = 1,\dots,5: \quad \sum_{\mathrm{i,j}=1}^{6} \tilde{\boldsymbol{\psi}}_{\mathrm{i}} \cdot [\mathrm{i}x_{\mathrm{a}}^{\mathrm{r}}]_{\mathrm{ij}} \left\langle \psi_{\mathrm{j}} \right\rangle_{\mathrm{U(3)}} = 0\\
\Rightarrow \quad &\tilde{\boldsymbol{\psi}} = \left[\sqrt{2n} + \tilde{\psi}_1, 0, 0, 0, 0, 0\right]
\end{aligned}
\tag{4.1.16}
$$

This choice of parameterization ensures that the ground state expectation value of all of the fields vanish and therefore the resulting Hamiltonian will be the sum of the constant ground state energy and operator terms with vanishing expectation value. By inserting the relation

$$
\boldsymbol{\psi} = \exp\{\mathrm{i}\mathbf{X}_{\mathrm{a}}\pi_{\mathrm{a}}\}\tilde{\boldsymbol{\psi}} \tag{4.1.17}
$$

into (4.1.4) and expanding to quadratic order in the fields we find that the free part $\mathcal{L}_0$ of the Lagrangian has the following structure:

$$
\begin{aligned}
&\mathcal{L}_0 = \sum_{\mathrm{a}=1}^{5} N_{\mathrm{a}}\dot{\pi}_{\mathrm{a}}\tilde{\psi}_1 + \sum_{\mathrm{a,b}=1}^{5} h_{\mathrm{ab}}\dot{\pi}_{\mathrm{a}}\pi_{\mathrm{b}} - \frac{1}{2}\sum_{\mathrm{a,b}=1}^{5} g_{\mathrm{ab}}\nabla\pi_{\mathrm{a}}\cdot\nabla\pi_{\mathrm{b}} - \frac{1}{2}\left(M\tilde{\psi}_1\tilde{\psi}_1 + G\nabla\tilde{\psi}_1\cdot\nabla\tilde{\psi}_1\right)\\
&N = \begin{bmatrix} 0\\0\\0\\0\\-\sqrt{2n} \end{bmatrix}, \quad g = \begin{bmatrix} \frac{n}{M}&0&0&0&0\\ 0&\frac{n}{M}&0&0&0\\ 0&0&\frac{n}{M}&0&0\\ 0&0&0&\frac{n}{M}&0\\ 0&0&0&0&\frac{n}{M} \end{bmatrix}, \quad h = \begin{bmatrix} 0&n&0&0&0\\ -n&0&0&0&0\\ 0&0&0&n&0\\ 0&0&-n&0&0\\ 0&0&0&0&0 \end{bmatrix}\\
&G = \frac{1}{2M}, \quad M = 2gn
\end{aligned}
\tag{4.1.18}
$$

From this it follows that the canonical momenta are just linear combinations of the field variables and therefore we have six constraints which can be expressed as linear combinations of fields and momenta. After defining a vector of all the

variables

$$\boldsymbol{V} = (\pi_1, \ldots, \pi_5, \tilde{\psi}_1, P_1, \ldots, P_6) \tag{4.1.19}$$

we can write

$$\begin{aligned} \varphi_{\mathrm{i}} &= \sum_{\mathrm{n}} [C]_{\mathrm{in}} V_{\mathrm{n}} \quad , \\ [C] &= \begin{bmatrix} 0 & n & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -n & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2n} & 0 & 0 & 0 & 0 & -1 & 0 \\ -n & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{4.1.20}$$

It is now possible to choose six new constraints which are equivalent to the old ones but simultaneously form a set of three canonically conjugate pairs of variables. First we find the Poisson brackets of the old constraints:

$$[\{\varphi_{\mathrm{i}}, \varphi_{\mathrm{j}}\}] = [C] S^{(12)} [C]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2n} \\ 2n & 0 & 0 & 0 & 0 & 0 \\ 0 & -2n & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2n} & 0 & 0 & 0 \end{bmatrix} \tag{4.1.21}$$

Solving the equation $[t][\{\varphi_{\mathrm{i}}, \varphi_{\mathrm{j}}\}][t]^{\mathrm{T}} = S^{(6)}$ for the transformation matrix $[t]$ and then calculating $[t][C]$ one finds that one of many possible choices is the following one:

$$\begin{aligned} \varphi'_{\mathrm{i}} &= \sum_{\mathrm{n}} [C']_{\mathrm{in}} V_{\mathrm{n}} \quad , \\ [C'] &= \begin{bmatrix} 0 & n & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -n & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2n} & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2n}} \end{bmatrix} \end{aligned} \tag{4.1.22}$$

This forms the basis for a canonical transformation with the special properties described in chapter 3.2. The general form of this canonical transformation will be

$$[\mathcal{C}] = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & n & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -n & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2n} & 0 & 0 & 0 & 0 & -1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2n} & 0 & 00 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2n}} \end{bmatrix} \tag{4.1.23}$$

The *-entries need to be determined via the condition

$$[\mathcal{C}]S^{(12)}[\mathcal{C}]^{\mathrm{T}} = S^{(12)} \quad . \tag{4.1.24}$$

The solution is not unique of course and in fact the solution space is so large that it seems impractical to compute the most general solution. By making some educated guess regarding the relevant entries of this matrix we find that one particular solution is

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{n} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & n & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -n & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2n} & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & \frac{n}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{n}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2n}} \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2n} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2n} & 0 & 00 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2n}}
\end{bmatrix} \tag{4.1.25}$$

The defining property of this matrix implies that the inverse is easily calculated as

$$[\mathcal{C}]^{-1} = -S^{(12)}[\mathcal{C}]^{\mathrm{T}}S^{(12)} \quad . \tag{4.1.26}$$

The relation between the old variables V and the new ones V' is given by

$$V_{\mathrm{i}} = \sum_{\mathrm{j}} [\mathcal{C}]^{-1}_{\mathrm{ij}} V'_{\mathrm{j}} \tag{4.1.27}$$

We can now re-express the free part of the Hamiltonian in terms of the primed coordinates and momenta and set

$$\psi'_4 = \psi'_5 = \psi'_6 = P'_4 = P'_5 = P'_6 = 0 \quad , \tag{4.1.28}$$

as this is the form the constraints take in these new coordinates and the special transformation that we constructed ensures that using these constraints does not affect any results one could calculate from the Hamiltonian. The naive free Hamiltonian resulting from (4.1.18) is

$$H_0 = \int_V \mathrm{d}^3x \ \sum_{\mathrm{i,j}} \left(\frac{1}{2} M_{\mathrm{ij}} \tilde{\psi}_{\mathrm{i}} \tilde{\psi}_{\mathrm{j}} + \frac{1}{2} G_{\mathrm{ij}} \nabla \tilde{\psi}_{\mathrm{i}} \cdot \nabla \tilde{\psi}_{\mathrm{j}} \right) + \frac{1}{2} \sum_{\mathrm{a,b}} \mathrm{g}_{\mathrm{ab}} \nabla \pi_{\mathrm{a}} \cdot \nabla \pi_{\mathrm{b}} \tag{4.1.29}$$

After defining Fourier components of fields as

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \exp\{\mathrm{i}\mathbf{k}\cdot\mathbf{x}\}\hat{\psi}(\mathbf{k}) \quad , \tag{4.1.30}$$

which satisfy the commutation relations

$$\{\hat{\psi}_{\mathrm{i}}(\mathbf{k}), \hat{P}_{\mathrm{j}}(\mathbf{k}')\} = \frac{1}{V}\delta_{\mathrm{i,j}}\delta_{\mathbf{k},-\mathbf{k}'} \quad , \tag{4.1.31}$$

we find the Hamiltonian in terms of these to be

$$H_0 = V\sum_{\mathbf{k}}\sum_{\mathrm{i,j}}\left(\frac{1}{2}M_{\mathrm{ij}}\hat{\tilde{\psi}}_{\mathrm{i}}(\mathbf{k})\hat{\tilde{\psi}}_{\mathrm{j}}(-\mathbf{k}) + \frac{1}{2}G_{\mathrm{ij}}|\mathbf{k}|^2\hat{\tilde{\psi}}_{\mathrm{i}}(\mathbf{k})\hat{\tilde{\psi}}_{\mathrm{j}}(-\mathbf{k})\right) + \frac{1}{2}\sum_{\mathrm{a,b}}\mathrm{g}_{\mathrm{ab}}|\mathbf{k}|^2\hat{\pi}_{\mathrm{a}}(\mathbf{k})\hat{\pi}_{\mathrm{b}}(-\mathbf{k}). \tag{4.1.32}$$

After changing to the primed variables we can introduce creation and annihilation operators

$$a_{\mathrm{i}}^{\dagger}(\mathbf{k}) = \sqrt{\frac{V}{2}}\left[\hat{\psi}'_{\mathrm{i}}(-\mathbf{k}) - \mathrm{i}\hat{P}'_{\mathrm{i}}(-\mathbf{k})\right], \quad a_{\mathrm{i}}(\mathbf{k}) = \sqrt{\frac{V}{2}}\left[\hat{\psi}'_{\mathrm{i}}(\mathbf{k}) + \mathrm{i}\hat{P}'_{\mathrm{i}}(\mathbf{k})\right] \quad . \tag{4.1.33}$$

These satisfy the commutation relations

$$\{a_{\mathrm{i}}(\mathbf{k}), a_{\mathrm{j}}^{\dagger}(\mathbf{k}')\} = \delta_{\mathrm{i,j}}\delta_{\mathbf{k},\mathbf{k}'} \quad . \tag{4.1.34}$$

After replacing The canonical fields and momenta with these creation and annihilation operators, the Hamiltonian takes the form

$$\begin{aligned} H_0 &= \sum_{i,j,\mathbf{k}} A_{\mathrm{i}}^{\dagger}(\mathbf{k})[\mathcal{H}]_{\mathrm{i,j}}A_{\mathrm{j}}(\mathbf{k}) \quad , \\ A(\mathbf{k}) &= \left(a_1^{\dagger}(\mathbf{k}), a_1(-\mathbf{k}), a_2^{\dagger}(\mathbf{k}), a_2(-\mathbf{k}), a_3^{\dagger}(\mathbf{k}), a_3(-\mathbf{k})\right) \quad . \end{aligned} \tag{4.1.35}$$

The quadratic form $[\mathcal{H}]$ can be diagonalized by a bosonic Bogoliubov transformation which preserves the commutation relations. The resulting spectrum consists of three dispersion relations which can be expressed using $\varepsilon(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2M}$ as

$$\begin{aligned} \omega_1(\mathbf{k}) &= \varepsilon(\mathbf{k}) \quad , \\ \omega_2(\mathbf{k}) &= \varepsilon(\mathbf{k}) \quad , \\ \omega_3(\mathbf{k}) &= \sqrt{\varepsilon(\mathbf{k})(\varepsilon(\mathbf{k}) + 2gn)} \quad . \end{aligned} \tag{4.1.36}$$

And the free Hamiltonian takes the form

$$H_0 = \sum_{\mathbf{k},\mathrm{i}} \omega_{\mathrm{i}}(\mathbf{k}) a_{\mathrm{i}}^{\dagger}(\mathbf{k}) a_{\mathrm{i}}(\mathbf{k}) \tag{4.1.37}$$

Contrary to the naive view that the massive fields should give rise to gapped modes and the Goldstone fields should give rise to gapless modes, we see that in this case they all combine via the elimination of constraints to give three gapless modes. It is therefore a priori not clear what kind of spectrum one should expect. However, as we will see in the next section, we get the same result by just taking into account the Goldstone fields but using the most general form of the Lagrangian based on the symmetries. It will turn out that some of the additional terms in the quadratic part of the Lagrangian are precisely those which are generated by integrating out the massive field. It is in this sense that the general claim of the low energy effective theory being the corresponding non-linear σ-model remains valid for the U(N) system.

4.1.3 General Coset Construction in the Symmetry Breaking Ground State

The general construction of a low energy Lagrangian based on the symmetry breaking pattern starts with the Lie algebra of the symmetry group U(3), which in this case is broken to the subgroup U(2). Following the notation of subsection 2.1 we have $x_1 = x_1^r$, ... , $x_5 = x_5^r$, $t_6 = t_6^r$, ... , $t_9 = t_9^r$. In order to write down the general effective Lagrangian in terms of Goldstone fields, i.e. the non-linear σ-model with the target space U(3)/U(2), we need to determine the coefficients in (2.2.1) via the conditions (2.2.2) and (2.2.3). Keeping in mind that only the symmetric part of $g^{t/\mathbf{x}}$ contributes to $\mathcal{L}$, one finds that

$$\vec{v} = (0, 0, 0, 0, v_5) \tag{4.1.38}$$

and

$$\left[g^{t/\mathbf{x}}\right] = \begin{bmatrix} g_{11}^{t/\mathbf{x}} & 0 & 0 & 0 & 0 \\ 0 & g_{11}^{t/\mathbf{x}} & 0 & 0 & 0 \\ 0 & 0 & g_{11}^{t/\mathbf{x}} & 0 & 0 \\ 0 & 0 & 0 & g_{11}^{t/\mathbf{x}} & 0 \\ 0 & 0 & 0 & 0 & g_{55}^{t/\mathbf{x}} \end{bmatrix} \quad . \tag{4.1.39}$$

The Lagrangian therefore has the structure

$$\begin{aligned} \mathcal{L} = v_5 \sum_a D_{5a}\dot{\pi}_a + \sum_{a,b}\left[\frac{g_{11}^t}{2}\left(D_{1a}D_{1b} + D_{2a}D_{2b} + D_{3a}D_{3b} + D_{4a}D_{4b}\right) + \frac{g_{55}^t}{2}D_{5a}D_{5b}\right]\dot{\pi}_a\dot{\pi}_b \\ + \sum_{a,b}\left[\frac{g_{11}^{\mathbf{x}}}{2}\left(D_{1a}D_{1b} + D_{2a}D_{2b} + D_{3a}D_{3b} + D_{4a}D_{4b}\right) + \frac{g_{55}^{\mathbf{x}}}{2}D_{5a}D_{5b}\right]\nabla\pi_a \cdot \nabla\pi_b + \dots \end{aligned} \tag{4.1.40}$$

In this case the terms involving the covariant derivative of D only give total derivatives which do not contribute to the Lagrangian density. The structure constants of the set of generators provide the power series of D_{ab} and by collecting all of the terms of order two in powers of Goldstone-fields we obtain the free Lagrangian density

$$\begin{aligned} \mathcal{L}_0 = v_5\dot{\pi}_5 + v_5\left[(\pi_1\dot{\pi}_2 - \pi_2\dot{\pi}_1) + (\pi_3\dot{\pi}_4 - \pi_4\dot{\pi}_3)\right] + \frac{g_{55}^t}{2}\dot{\pi}_5^2 + \frac{g_{11}^t}{2}\left[\dot{\pi}_1^2 + \dot{\pi}_2^2 + \dot{\pi}_3^2 + \dot{\pi}_4^2\right] \\ - \frac{g_{55}^{\mathbf{x}}}{2}\nabla\pi_5 \cdot \nabla\pi_5 - \frac{g_{11}^{\mathbf{x}}}{2}\left[\nabla\pi_1 \cdot \nabla\pi_1 + \nabla\pi_2 \cdot \nabla\pi_2 + \nabla\pi_3 \cdot \nabla\pi_3 + \nabla\pi_4 \cdot \nabla\pi_4\right] \end{aligned} \tag{4.1.41}$$

The very first term can be dropped, being a total derivative. In the presence of first order time derivative terms for a given field the second order derivative terms for that field can be treated as perturbative interactions in the low energy limit and so

we see that we can write the free Lagrangian as

$$\begin{aligned}\mathcal{L}_0 =& \mathrm{v}_5 \left[(\pi_1\dot{\pi}_2 - \pi_2\dot{\pi}_1) + (\pi_3\dot{\pi}_4 - \pi_4\dot{\pi}_3)\right] + \frac{g^t_{55}}{2}\dot{\pi}_5^2 \\ &- \frac{g^{\mathbf{x}}_{55}}{2}\nabla\pi_5\cdot\nabla\pi_5 - \frac{g^{\mathbf{x}}_{11}}{2}\left[\nabla\pi_1\cdot\nabla\pi_1 + \nabla\pi_2\cdot\nabla\pi_2 + \nabla\pi_3\cdot\nabla\pi_3 + \nabla\pi_4\cdot\nabla\pi_4\right]\end{aligned} \tag{4.1.42}$$

We can already see at this stage that there will be one mode corresponding to π_5 which should have a linear dispersion relation while the fields π_1, π_2 and π_3, π_4 pair up to form two modes with quadratic dispersion relations. The definition of the canonical momenta gives an invertible relation between $\dot{\pi}_5$ and P_5 and four constraints:

$$\begin{aligned}P_5 &= g^t_{55}\dot{\pi}_5 \\ \vec{\varphi} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 & 0 & -\frac{1}{\sqrt{\mathrm{v}_5}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{\mathrm{v}_5}} \\ 1 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{\mathrm{v}_5}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{\sqrt{\mathrm{v}_5}} & 0 \end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}\end{aligned} \tag{4.1.43}$$

In this form the constraints form two pairs of canonical variables and we therefore have four second class constraints and no additional ones are generated by the consistency conditions. The naive Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2g^t_{55}}P_5^2 + \frac{g^{\mathbf{x}}_{55}}{2}\nabla\pi_5\cdot\nabla\pi_5 + \frac{g^{\mathbf{x}}_{11}}{2}\left[\nabla\pi_1\cdot\nabla\pi_1 + \nabla\pi_2\cdot\nabla\pi_2 + \nabla\pi_3\cdot\nabla\pi_3 + \nabla\pi_4\cdot\nabla\pi_4\right] \tag{4.1.44}$$

and the constraints can be eliminated by performing the following canonical transformation:

$$\begin{bmatrix}\pi_1'\\ \pi_2'\\ \pi_3'\\ \pi_4'\\ \pi_5'\\ P_1'\\ P_2'\\ P_3'\\ P_4'\\ P_5'\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mathrm{v}_5} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mathrm{v}_5} & 0 \\ 0 & -\sqrt{\frac{\mathrm{v}_5}{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2\mathrm{v}_5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\mathrm{v}_5}{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2\mathrm{v}_5}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{v}_5}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{v}_5}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \sqrt{\frac{\mathrm{v}_5}{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2\mathrm{v}_5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\mathrm{v}_5}{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2\mathrm{v}_5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix}, \tag{4.1.45}$$

which transform the constraints into the conditions

$$\pi_3' = \pi_4' = P_3' = P_4' = 0 \quad . \tag{4.1.46}$$

The Hamiltonian in terms of Fourier components therefore reads

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\Bigg[&\frac{g_{11}^{\mathbf{x}}}{4}k^2\left(\hat{\pi}_1'(-\mathbf{k})\hat{\pi}_1'(\mathbf{k}) + \hat{\pi}_2'(-\mathbf{k})\hat{\pi}_2'(\mathbf{k})\right) + g_{55}^{\mathbf{x}}k^2\hat{\pi}_5'(-\mathbf{k})\hat{\pi}_5'(\mathbf{k}) \\ &+\frac{g_{11}^{\mathbf{x}}}{v_5}k^2\left(\hat{P}_1'(-\mathbf{k})\hat{P}_1'(\mathbf{k}) + \hat{P}_2'(-\mathbf{k})\hat{P}_2'(\mathbf{k})\right) + g_{55}^{t}\hat{P}_5'(-\mathbf{k})\hat{P}_5'(\mathbf{k})\Bigg]\end{aligned} \tag{4.1.47}$$

After replacing the fields and momenta with creation and annihilation operators the Hamiltonian can be diagonalized via a bosonic Bogoliubov transformation and the spectrum one obtains is

$$\omega_{1,2}(\mathbf{k}) = k^2\sqrt{\frac{{g_{11}^{\mathbf{x}}}^2}{4v_5^2}} \quad , \quad \omega_3(\mathbf{k}) = |k|\sqrt{g_{55}^{t}g_{55}^{\mathbf{x}}} \quad . \tag{4.1.48}$$

The free Hamiltonian takes the standard form

$$\mathcal{H}_0 = \sum_{\mathbf{k}}\sum_{i=1}^{3}\omega_i(\mathbf{k})a_i^{\dagger}(\mathbf{k})a_i(\mathbf{k}) \tag{4.1.49}$$

4.1.4 Interactions

The interactions of lowest order in the Goldstone fields are just the quadratic term that we neglected in the free Lagrangian:

$$\mathcal{L}_{\text{int},\mathcal{O}(2)} = \frac{g_{11}^{t}}{2}\left[\dot{\pi}_1^2 + \dot{\pi}_2^2 + \dot{\pi}_3^2 + \dot{\pi}_4^2\right] \tag{4.1.50}$$

Since the dispersion of the modes associated with π_1 to π_4 is quadratic in $|\mathbf{k}|$ this interaction is proportional to $|\mathbf{k}|^4$. There is however an interaction of lower order in $|\mathbf{k}|$. By expanding (4.1.40) further we find the interaction terms of third order in the Goldstone fields to be

$$\begin{aligned}\mathcal{L}_{\text{int},O(3)} =& g_{55}^{t}\left[\pi_1\dot{\pi}_2 - \pi_2\dot{\pi}_1 + \pi_4\dot{\pi}_3 - \pi_3\dot{\pi}_4\right]\dot{\pi}_5 \\ &- g_{55}^{\mathbf{x}}\left[\pi_1\nabla\pi_2 - \pi_2\nabla\pi_1 + \pi_3\nabla\pi_4 - \pi_4\nabla\pi_3\right]\cdot\nabla\pi_5 \quad .\end{aligned} \tag{4.1.51}$$

After applying the canonical transformation, replacing fields with creation and annihilation operators and then applying the bosonic Bogoliubov transformation

we obtain the interactions

$$\begin{aligned}\mathcal{H}_{\text{int},\mathcal{O}(3)} = \frac{\mathrm{i}}{\sqrt{V}} \frac{g^{\mathrm{x}}_{55}}{2\sqrt{2}\left(g^{\mathrm{x}}_{55} g^{t}_{55}\right)^{\frac{1}{4}} \mathrm{v}_5} \sum_{\mathbf{k}'} \frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{k}')}{\sqrt{|\mathbf{k}+\mathbf{k}'|}} \left[a_3^\dagger(\mathbf{k}+\mathbf{k}') + a_3(-(\mathbf{k}+\mathbf{k}'))\right] \\ \cdot \left[a_1^\dagger(-\mathbf{k})a_1(\mathbf{k}') - a_1^\dagger(-\mathbf{k}')a_1(\mathbf{k}) + a_2^\dagger(-\mathbf{k})a_2(\mathbf{k}') - a_2^\dagger(-\mathbf{k}')a_2(\mathbf{k})\right] \\ + \frac{\mathrm{i}}{\sqrt{V}} \frac{g^{\mathrm{x}}_{11}}{4\sqrt{2}\mathrm{v}_5^2} \left(g^{\mathrm{x}}_{55} g^{t}_{55}\right)^{\frac{1}{4}} \sum_{\mathbf{k}'} |\mathbf{k}|^2 \sqrt{|\mathbf{k}+\mathbf{k}'|} \left[a_3^\dagger(\mathbf{k}+\mathbf{k}') - a_3(-(\mathbf{k}+\mathbf{k}'))\right] \\ \cdot \left[a_1^\dagger(-\mathbf{k})a_1(\mathbf{k}') + a_1^\dagger(-\mathbf{k}')a_1(\mathbf{k}) + a_2^\dagger(-\mathbf{k})a_2(\mathbf{k}') + a_2^\dagger(-\mathbf{k}')a_2(\mathbf{k})\right] \quad ,\end{aligned} \tag{4.1.52}$$

where V is the volume of the system. The first term is of order $|\mathbf{k}|^{\frac{3}{2}}$ and the second one of order $|\mathbf{k}|^{\frac{5}{2}}$.
The interactions to fourth order in the fields take the form

$$\mathcal{L}_{\text{int},\mathcal{O}(4)} = \pi_{\mathrm{i}}\pi_{\mathrm{j}}\dot{\pi}_{\mathrm{m}}\dot{\pi}_{\mathrm{n}} \quad \text{-terms} \quad + \quad \pi_{\mathrm{i}}\pi_{\mathrm{j}}\nabla\pi_{\mathrm{m}}\cdot\nabla\pi_{\mathrm{n}} \quad \text{-terms} \quad , \tag{4.1.53}$$

where the indices i, j, m, n run over $1, \ldots, 4$. All of these terms contribute interactions that are proportional to $|\mathbf{k}|^4$ but do not involve the Bogoliubov mode $a_3^\dagger$.

4.1.5 Matching Coefficients

After writing down the path integral of the Lagrangian written in terms of Goldstone- and massive fields we can integrate out the latter and obtain a Lagrangian purely in terms of Goldstone fields which we can then compare to the one obtained via the general construction method. This allows us to match the so far arbitrary coefficients.

$$
\begin{aligned}
&\int \mathcal{D}(\pi)\mathcal{D}(\tilde{\psi}) \exp\left[\int_x \mathcal{L}'(\pi,\tilde{\psi})\right] \\
&= \int \mathcal{D}(\pi) \left\{ \exp\left[\int_x \sum_{a,b=1}^{5} h_{ab}\dot{\pi}_a\pi_b - \frac{1}{2}g_{ab}\nabla\pi_a\cdot\nabla\pi_b\right] \right. \\
&\qquad \left. \cdot \int \mathcal{D}(\tilde{\psi}) \exp\left[\int_x -\frac{1}{2}\left(2gn\tilde{\psi}_1^2 + \frac{1}{2M}\nabla\tilde{\psi}_1\cdot\nabla\tilde{\psi}_1\right) - \sqrt{2n}\dot{\pi}_5\tilde{\psi}_1\right]\right\} \\
&= \int \mathcal{D}(\hat{\pi}) \left\{ \exp\left[\int_t V \sum_{\mathbf{k}} \sum_{a,b=1}^{5} h_{ab}(\dot{\hat{\pi}}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}} - \frac{1}{2}g_{ab}|\mathbf{k}|^2(\hat{\pi}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}}\right] \right. \\
&\qquad \left. \cdot \int \mathcal{D}(\hat{\tilde{\psi}}) \exp\left[\int_t V \sum_{\mathbf{k},\mathbf{k}'} -\frac{1}{2}(\hat{\tilde{\psi}}_1)_{\mathbf{k}}\left[\left(2gn + \frac{|\mathbf{k}|^2}{2M}\right)\delta_{\mathbf{k},-\mathbf{k}'}\right](\hat{\tilde{\psi}}_1)_{\mathbf{k}'} - \delta_{\mathbf{k},-\mathbf{k}'}\sqrt{2n}(\dot{\hat{\pi}}_5)_{\mathbf{k}}(\hat{\tilde{\psi}}_1)_{\mathbf{k}'}\right]\right\} \\
&\propto \int \mathcal{D}(\hat{\pi}) \left\{ \exp\left[\int_t V \sum_{\mathbf{k}} \sum_{a,b=1}^{5} h_{ab}(\dot{\hat{\pi}}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}} - \frac{1}{2}g_{ab}|\mathbf{k}|^2(\hat{\pi}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}}\right] \right. \\
&\qquad \left. \cdot \exp\left[\int_t V \sum_{\mathbf{k}} \frac{n}{2gn + \frac{|\mathbf{k}|^2}{2M}}(\dot{\hat{\pi}}_5)_{\mathbf{k}}(\dot{\hat{\pi}}_5)_{-\mathbf{k}}\right]\right\}
\end{aligned}
\tag{4.1.54}
$$

In the first step the action has been separated into terms which do not involve $\tilde{\psi}_1$ and those which do. In the second step the integral over space was replaced by a sum over Fourier components. After completing the square in the path integral over $\tilde{\psi}_1$ this integral can be evaluated where an overall constant factor has been dropped since it cancels in the calculation of any correlation functions. By expanding the factor $\frac{n}{2gn+\frac{|\mathbf{k}|^2}{2M}}$ in powers of $|\mathbf{k}|$ we can write the free part of the action as

$$
\begin{aligned}
\mathcal{L}_0(\pi) &= \int_t V \sum_{\mathbf{k}} \sum_{a,b=1}^{5} \left(h_{ab}(\dot{\hat{\pi}}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}} - \frac{1}{2}g_{ab}|\mathbf{k}|^2(\hat{\pi}_a)_{\mathbf{k}}(\hat{\pi}_b)_{-\mathbf{k}}\right) + \frac{1}{2g}(\dot{\hat{\pi}}_5)_{\mathbf{k}}(\dot{\hat{\pi}}_5)_{-\mathbf{k}} \\
&= \int_x \sum_{a,b=1}^{5} h_{ab}\dot{\pi}_a\pi_b + \frac{1}{2g}\dot{\pi}_5^2 - \frac{1}{2}\sum_{a,b=1}^{5} g_{ab}\nabla\pi_a\cdot\nabla\pi_b
\end{aligned}
\tag{4.1.55}
$$

By comparing (4.1.18) and (4.1.41) we find that

$$v_5 = -n, \quad g^t_{11} = 0, \quad g^t_{55} = 1/g, \quad g^{\mathbf{x}}_{11} = n/M, \quad g^{\mathbf{x}}_{55} = n/M \quad . \tag{4.1.56}$$

With these values the dispersion relations (4.1.48) indeed match up with (4.1.36). If we define

$$v_5(\alpha) \coloneqq \alpha \cdot (-n), \quad g^t_{55}(\gamma) = \gamma \cdot \frac{1}{g}, \quad g^{\mathbf{x}}_{11}(\sigma) = \sigma \cdot \frac{n}{M}, \quad g^{\mathbf{x}}_{55}(\lambda) = \lambda \cdot \frac{n}{M} \tag{4.1.57}$$

then the spectrum (4.1.48) is invariant if we keep $\sigma = \alpha$ and $\gamma \cdot \lambda = 1$ and the interactions (4.1.52) acquire the pre-factors $\frac{\lambda}{\alpha}$ and $\frac{1}{\alpha}$ respectively. This shows that the relative proportions of the terms in the General Lagrangian is still arbitrary after fixing the dispersion relations.

4.2 Spin 1 - Lagrangian with Quadratic Zeeman-Effect

We postulate that the correct Lagrangian density written in terms of three complex fields $(\psi_1, \psi_2, \psi_3) \in \mathbb{C}^3$ takes the form

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{3}\left[\frac{1}{2}i(\psi_i^*\partial_t\psi_i - \psi_i\partial_t\psi_i^*) - \frac{1}{2m}\nabla\psi_i^*\cdot\nabla\psi_i\right] - \frac{c_0}{2}\left(\sum_{i=1}^{3}\psi_i^*\psi_i\right)^2 \\ &- \sum_{i,j,l,m=1}^{3}\frac{c_1}{2}\left(\psi_i^*\vec{\mathbf{f}}_{ij}\psi_j\right)\cdot\left(\psi_l^*\vec{\mathbf{f}}_{lm}\psi_m\right) - q\left(\psi_1^*\psi_1 + \psi_3^*\psi_3\right) + \sum_{i=1}^{3}\mu_i\psi_i^*\psi_i\end{aligned} \tag{4.2.1}$$

The matrices $\mathbf{f}$ appearing here denote the three spin generators in the spin-1 representation:

$$\mathbf{f}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} \quad , \quad \mathbf{f}_2 = \frac{i}{\sqrt{2}}\begin{bmatrix}0&-1&0\\1&0&-1\\0&1&0\end{bmatrix} \quad , \quad \mathbf{f}_3 = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} \tag{4.2.2}$$

Rotations are generated by these matrices in the sense that $\psi_a \to (\exp\left[i\vec{\theta}\cdot\vec{\mathbf{f}}\right])_{ab}\psi_b$. The last term in (4.2.1) describes the quadratic Zeeman effect and if this term was absent then $\mathcal{L}$ would be completely symmetric under SO(3). The presence of this term however means that this symmetry is explicitly broken to SO(2), namely rotations about the z-axis which are generated by $\mathbf{f}_3$. There is also a U(1) symmetry which is obvious in the formulation using complex fields and whose generator is just the unit matrix $\mathbb{1}$. Again, one needs the last term in order to fix the total particle density but now the different spin components will have different chemical potentials and we need to be clear on whether this term breaks any symmetry explicitly, which in fact it does not, as can be seen from the fact that neither $\mathbb{1}$ nor $\mathbf{f}_3$ mixes the different components.

$\mathcal{L}$ is a real function and can be written in terms of six real fields, namely the real and imaginary parts of the ψ_a. If we define $\psi_a = (\psi_{a1} + i\psi_{a2})/\sqrt{2}$ then we get

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{3}\left[\frac{1}{2}(\psi_{i2}\partial_t\psi_{i1} - \psi_{i1}\partial_t\psi_{i2}) - \frac{1}{4M}\left(\nabla\psi_{i1}\cdot\nabla\psi_{i1} + \nabla\psi_{i2}\cdot\nabla\psi_{i2}\right)\right] \\ &- \frac{c_0}{8}\left(\sum_{i=1}^{3}\left(\psi_{i1}^2 + \psi_{i2}^2\right)\right)^2 - \sum_{i,j=1}^{3}\frac{c_1}{8}\left|\psi_{i1}\vec{\mathbf{f}}_{ij}\psi_{j1} + \psi_{i2}\vec{\mathbf{f}}_{ij}\psi_{j2} + i\psi_{i1}\vec{\mathbf{f}}_{ij}\psi_{j2} - i\psi_{i2}\vec{\mathbf{f}}_{ij}\psi_{j1}\right|^2 \\ &- \frac{q}{2}\left(\psi_{11}^2 + \psi_{12}^2 + \psi_{31}^2 + \psi_{32}^2\right) + \frac{\mu}{2}\sum_{i=1}^{3}\left(\psi_{i1}^2 + \psi_{i2}^2\right)\end{aligned} \tag{4.2.3}$$

Furthermore, by introducing the vector

$$\Psi = [\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}, \psi_{31}, \psi_{32}] \quad , \tag{4.2.4}$$

as well as the real form of the spin matrices **F** we can simplify the notation and write

$$\mathcal{L} = \frac{1}{2}\left[\Psi[\mathrm{i}\mathbf{X}_2]\partial_t\Psi - \frac{1}{2M}\nabla\Psi\cdot\nabla\Psi - \frac{c_0}{4}(\Psi\cdot\Psi)^2 - \frac{c_1}{4}\left(\Psi\vec{\mathbf{F}}\Psi\right)\cdot\left(\Psi\vec{\mathbf{F}}\Psi\right) - q\Psi\mathbf{F}_3^2\Psi + \mu\Psi\cdot\Psi\right] \tag{4.2.5}$$

The real matrices **F** are obtained from the complex **f** by matching the results of either applying **f** to the complex fields and then splitting the vector into real and imaginary components or applying the real matrices **F** to the real version of the field vector. One finds

$$\mathbf{F}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix}, \quad \mathbf{F}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0&0&0&1&0&0\\ 0&0&-1&0&0&0\\ 0&-1&0&0&0&1\\ 1&0&0&0&-1&0\\ 0&0&0&-1&0&0\\ 0&0&1&0&0&0 \end{bmatrix}, \quad \mathbf{F}_3 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{bmatrix} \tag{4.2.6}$$

Similarly, one finds the real version of the symmetry generators. Here it has to be kept in mind that the change in the complex field components is actually proportional to $\mathrm{i}\mathbf{f}_{ab}\psi_b$ and so one needs to apply this method of matching results to i**f** instead of **f**. Also since we follow the convention of always writing finite transformations as $\exp[\mathrm{i}\mathbf{X}]$ we need to multiply the real matrix version of the generators by $(-\mathrm{i})$. We thereby obtain the two symmetry generators in the real-field formulation as

$$\mathbf{X}_1 = \begin{bmatrix} 0&\mathrm{i}&0&0&0&0\\ -\mathrm{i}&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-\mathrm{i}\\ 0&0&0&0&\mathrm{i}&0 \end{bmatrix} \quad , \quad \mathbf{X}_2 = \begin{bmatrix} 0&\mathrm{i}&0&0&0&0\\ -\mathrm{i}&0&0&0&0&0\\ 0&0&0&\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&0&0&0\\ 0&0&0&0&0&\mathrm{i}\\ 0&0&0&0&-\mathrm{i}&0 \end{bmatrix} \quad . \tag{4.2.7}$$

Hamiltonian In The Fundamental Field Formulation

The quantization procedure is completely anologous to the one for the U(N) Lagrangian. The very first term in (4.2.3) can be exchanged for $\sum_{i=1}^{3}\psi_{i2}\partial_t\psi_{i1}$ by partial integration with respect to the time variable. The definition of the canonical momenta yields six constraints:

$$\varphi_{i,1} = \psi_{i,2} - P_{i,1} \quad , \quad \varphi_{i,2} = P_{i,2} \quad , \text{with} \quad i = 1,2,3 \tag{4.2.8}$$

By calculating their Poisson brackets we find that $M = S^{(6)}$, i.e. the symplectic metric and since $\det(S) = 1$ we have six second class constraints.Their Dirac brackets are again just equal to their Poisson brackets. The naive Hamiltonian

density reads

$$\begin{aligned}\mathcal{H} = &\sum_{i=1}^{3} \frac{1}{4m}(\nabla\psi_{i1}\cdot\nabla\psi_{i1} + \nabla P_{i1}\cdot\nabla P_{i1}) + \frac{c_0}{8}\left(\sum_{i=1}^{3}(\psi_{i1}^2 + P_{i1}^2)\right)^2 \\ &+ \sum_{i,j=1}^{3}\frac{c_1}{8}\left|\psi_{i1}\vec{\mathbf{f}}_{ij}\psi_{j1} + P_{j1}\vec{\mathbf{f}}_{ij}P_{j1} + i\psi_{i1}\vec{\mathbf{f}}_{ij}P_{j1} - iP_{i1}\vec{\mathbf{f}}_{ij}\psi_{j1}\right|^2 + q(\psi_{11}^2 + P_{11}^2 + \psi_{31}^2 + P_{31}^2) \\ &+ \mu\sum_{i=1}^{3}\left(\psi_{i1}^2 + P_{i1}^2\right) \quad ,\end{aligned} \tag{4.2.9}$$

where we have again used the constraints in order to replace half of the fields with momenta. The remaining fields and momenta again obey canonical commutation relations if we replace Dirac brackets by (-i) times the commutators and so we define the creation and annihilation operators $\Phi^\dagger$ and Φ just as in (4.1.7). Inserting these relations into (4.2.9) and using partial integration with respect to the spacial derivatives in the first term gives

$$\begin{aligned}\mathcal{H} = \sum_{i=1}^{3}\Phi_i^\dagger\left(-\frac{\nabla^2}{2m}\Phi_i\right) + \frac{c_0}{2}\left(\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i\right)^2 + \frac{c_1}{2}\left|\sum_{i,j=1}^{3}\Phi_i^\dagger\vec{\mathbf{f}}_{ij}\Phi_j\right|^2 \\ + q\left(\Phi_1^\dagger\Phi_1 + \Phi_3^\dagger\Phi_3\right) - \mu\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i \quad ,\end{aligned} \tag{4.2.10}$$

which is indeed the correct Hamiltonian density.

4.2.1 Explicit Parameterization in The Broken-Axisymmetry Phase

The method of stationary phase approximation allows one to determine the different local minima of the Hamiltonian and to identify the order parameters in each of the different phases. One of the solutions, referred to as the broken-axisymmetry phase (BA), is characterized by the ground state expectation values of the real fundamental fields being

$$\langle\Psi\rangle_{BA} = \left(\sqrt{n\left(\frac{1}{2}+Q\right)}, 0, \sqrt{n\,(1-2Q)}, 0, \frac{1}{2}\sqrt{n\left(\frac{1}{2}+Q\right)}, 0\right) \tag{4.2.11}$$

From here on we use

$$Q = \frac{q}{4nc_1} \quad , \quad \text{with} \qquad \left(-\frac{1}{2}\right) < Q < 0 \tag{4.2.12}$$

instead of q.

One simultaneusly determines the chemical potential to take the value

$$\mu = n\left(c_0 + c_1(1+2Q)\right) \quad . \tag{4.2.13}$$

This ground state breaks the SO(2)xU(1) symmetry completely since the equation $(\alpha \mathrm{i}\mathbf{X}_1 + \beta \mathrm{i}\mathbf{X}_2)\left\langle\Psi\right\rangle_{\mathrm{BA}} = 0$ only has the trivial solution $\alpha = \beta = 0$. We now determine the massive fields $\tilde{\psi}$ via the condition (2.1.5):

$$\begin{aligned} &\sum_{\mathrm{i,j}=1}^{6} \tilde{\boldsymbol{\psi}}_{\mathrm{i}} \cdot [\mathrm{i}\mathbf{X}_1]_{\mathrm{ij}} \left\langle \psi_{\mathrm{j}} \right\rangle_{\mathrm{BA}} = \sum_{\mathrm{i,j}=1}^{6} \tilde{\boldsymbol{\psi}}_{\mathrm{i}} \cdot [\mathrm{i}\mathbf{X}_2]_{\mathrm{ij}} \left\langle \psi_{\mathrm{j}} \right\rangle_{\mathrm{BA}} = 0 \\ \Rightarrow \quad &\tilde{\boldsymbol{\psi}} = \left[\sqrt{n\left(\frac{1}{2}+Q\right)} + \tilde{\psi}_1, \tilde{\psi}_4, \sqrt{n\left(1-2Q\right)} + \tilde{\psi}_2, -\sqrt{\frac{2+4Q}{1-2Q}}\tilde{\psi}_4, \sqrt{n\left(\frac{1}{2}+Q\right)} + \tilde{\psi}_3, \tilde{\psi}_4 \right] \end{aligned} \tag{4.2.14}$$

The choice of parameterization for the first, third and fifth component ensures that the ground state expectation value of every field appearing here vanishes and so we should end up with a Hamiltonian which is the sum of the ground state energy and terms with vanishing expectation values. By inserting the relation

$$\boldsymbol{\psi} = \exp\{\mathrm{i}\mathbf{X}_{\mathrm{a}} \pi_{\mathrm{a}}\} \tilde{\boldsymbol{\psi}} \tag{4.2.15}$$

into (4.2.3) and using the fact that the structure constants of the Lie algebra are trivial we can write the Lagrangian as

$$\begin{aligned} \mathcal{L} = \frac{1}{2} \Bigg[\tilde{\boldsymbol{\psi}} \cdot [\mathrm{i}\mathbf{X}_2] \dot{\tilde{\boldsymbol{\psi}}} + \sum_{\mathrm{a}=1}^{2} \dot{\pi}_{\mathrm{a}} \tilde{\boldsymbol{\psi}} \cdot [\mathrm{i}\mathbf{X}_2][\mathrm{i}\mathbf{X}_{\mathrm{a}}] \dot{\tilde{\boldsymbol{\psi}}} - \frac{1}{2M} \left(\nabla\tilde{\boldsymbol{\psi}} \cdot \nabla\tilde{\boldsymbol{\psi}} + \sum_{\mathrm{a,b}=1}^{2} \nabla\pi_{\mathrm{a}} \cdot \nabla\pi_{\mathrm{b}} \tilde{\boldsymbol{\psi}} \cdot [\mathbf{X}_{\mathrm{a}}][\mathbf{X}_{\mathrm{b}}] \tilde{\boldsymbol{\psi}} \right) \\ - \frac{c_0}{4} \left(\tilde{\boldsymbol{\psi}} \cdot \tilde{\boldsymbol{\psi}}\right)^2 - \frac{c_1}{4} \left(\tilde{\boldsymbol{\psi}} \vec{\mathbf{F}} \tilde{\boldsymbol{\psi}} \right) \cdot \left(\tilde{\boldsymbol{\psi}} \vec{\mathbf{F}} \tilde{\boldsymbol{\psi}} \right) - q \tilde{\boldsymbol{\psi}} \mathbf{F}_3^2 \tilde{\boldsymbol{\psi}} + \mu \tilde{\boldsymbol{\psi}} \cdot \tilde{\boldsymbol{\psi}} \Bigg] \end{aligned} \tag{4.2.16}$$

Expanding to quadratic order in the fields we find that the free part $\mathcal{L}_0$ of the

Lagrangian has the following structure:

$$\mathcal{L} = \sum_{a,i} N_{ai}\dot{\pi}_a\tilde{\psi}_i - \frac{1}{2}\sum_{a,b} g_{ab}\nabla\pi_a\cdot\nabla\pi_b + \sum_{i,j}\left(T_{ij}\tilde{\psi}_i\dot{\tilde{\psi}}_j - \frac{1}{2}M_{ij}\tilde{\psi}_i\tilde{\psi}_j - \frac{1}{2}G_{ij}\nabla\tilde{\psi}_i\cdot\nabla\tilde{\psi}_j\right)$$

$$N = \begin{bmatrix} -\sqrt{n(\frac{1}{2}+Q)} & 0 & \sqrt{n(\frac{1}{2}+Q)} & 0 \\ -\sqrt{n(\frac{1}{2}+Q)} & -\sqrt{n(1-2Q)} & -\sqrt{n(\frac{1}{2}+Q)} & 0 \end{bmatrix} \quad , \quad g = \begin{bmatrix} \frac{n(1+2Q)}{2M} & 0 \\ 0 & \frac{n}{M} \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \sqrt{\frac{1+2Q}{2-4Q}} \\ 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\sqrt{\frac{1+2Q}{2-4Q}} & \frac{1}{2} & 0 \end{bmatrix} \quad , \quad G = \begin{bmatrix} \frac{1}{2M} & 0 & 0 & 0 \\ 0 & \frac{1}{2M} & 0 & 0 \\ 0 & 0 & \frac{1}{2M} & 0 \\ 0 & 0 & 0 & \frac{1}{M(1-2Q)} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{n}{2}(c_0(1+2Q)+4c_1Q) & n(c_0+2c_1)\sqrt{\frac{1}{2}-2Q^2} & \frac{n}{2}(c_0(1+2Q)-4c_1Q) & n(c_0+2c_1) & 0 \\ n(c_0+2c_1)\sqrt{\frac{1}{2}-2Q^2} & nc_0(1-2Q) & n(c_0+2c_1)\sqrt{\frac{1}{2}-2Q^2} & 0 & \\ \frac{n}{2}(c_0(1+2Q)-4c_1Q) & n(c_0+2c_1) & n(c_0+2c_1)\sqrt{\frac{1}{2}-2Q^2} & \frac{n}{2}(c_0(1+2Q)+4c_1Q) & 0 \\ 0 & 0 & 0 & -\frac{8nc_1}{1-2Q} & \end{bmatrix} \tag{4.2.17}$$

From this it follows that the canonical momenta are just linear combinations of the field variables and therefore we have six constraints which can be expressed as linear combinations of fields and momenta. After defining a vector of all the variables

$$V = (\pi_1, \pi_2, \tilde{\psi}_1, \ldots, \tilde{\psi}_4, P_1, \ldots, P_6) \tag{4.2.18}$$

we can write

$$\varphi_i = \sum_n [C]_{in} V_n \quad ,$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+2Q}{2-4Q} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{n}{2}\sqrt{2+4Q} & 0 & -\frac{n}{2}\sqrt{2+4Q} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{n}{2}\sqrt{2+4Q} & \sqrt{n(1-2Q)} & \frac{n}{2}\sqrt{2+4Q} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\sqrt{\frac{1+2Q}{2-4Q}} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.2.19}$$

$$\varphi'_i = \sum_n [C']_{in} V_n \quad ,$$

$$[C'] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+2Q}{2-4Q} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{8}(7+2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{8}(1-2Q) & 0 & -\frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \\ 0 & 0 & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{4}(2Q-3) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & 0 & 0 & -\sqrt{\frac{1-2Q}{4n}} & 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{2}-2Q^2} \\ 0 & 0 & \frac{1}{8}(1-2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & -\frac{1}{8}(7+2Q) & 0 & \frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \end{bmatrix}$$

(4.2.20)

This forms the basis for a canonical transformation with the special properties described in chapter 3.2. The general form of this canonical transformation will be

$$[\mathcal{C}] = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+2Q}{2-4Q} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & -\frac{1}{8}(7+2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{8}(1-2Q) & 0 & -\frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \\ 0 & 0 & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{4}(2Q-3) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & 0 & 0 & -\sqrt{\frac{1-2Q}{4n}} & 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{2}-2Q^2} \\ 0 & 0 & \frac{1}{8}(1-2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & -\frac{1}{8}(7+2Q) & 0 & \frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \end{bmatrix} \tag{4.2.21}$$

The *-entries again need to be determined via the condition $[\mathcal{C}]S^{(12)}[\mathcal{C}]^{\mathrm{T}} = S^{(12)}$. By making some guesses regarding the relevant entries of this matrix we find that one particular solution is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{n(2+4Q)}} & 0 & -\frac{1}{\sqrt{n(2+4Q)}} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1+2Q}{8n}} & \sqrt{\frac{1-2Q}{4n}} & \sqrt{\frac{1+2Q}{8n}} & 0 \\ 0 & \sqrt{n(2+4Q)} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+2Q}{2-4Q} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\sqrt{n(2+4Q)} & -\frac{n(1+2Q)}{2\sqrt{n(1-2Q)}} & \frac{1}{4}\sqrt{n(2+4Q)} & 0 & 0 & 1 & 0 & 0 & 0 & -\sqrt{\frac{n}{2}(1+2Q)} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2}\sqrt{\frac{1+2Q}{2-4Q}} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{8}(7+2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{8}(1-2Q) & 0 & -\frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \\ 0 & 0 & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & \frac{1}{4}(2Q-3) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & 0 & 0 & -\sqrt{\frac{1-2Q}{4n}} & 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{2}-2Q^2} \\ 0 & 0 & \frac{1}{8}(1-2Q) & -\frac{1}{4}\sqrt{\frac{1}{2}-2Q^2} & -\frac{1}{8}(7+2Q) & 0 & \frac{1}{\sqrt{n(2+4Q)}} & -\sqrt{\frac{1+2Q}{8n}} & 0 & 0 & 0 & \frac{1}{4}(2Q-1) \end{bmatrix} \tag{4.2.22}$$

The relation between the old variables V and the new ones V' is given by

$$V_{\mathrm{i}} = \sum_{\mathrm{j}} [\mathcal{C}]^{-1}_{\mathrm{ij}} V'_{\mathrm{j}} \tag{4.2.23}$$

We can now re-express the free part of the Hamiltonian in terms of the primed coordinates and momenta and set $\psi'_4 = \psi'_5 = \psi'_6 = P'_4 = P'_5 = P'_6 = 0$ as this is the form the constraints take in these new coordinates and the special transformation that we constructed ensures that using these constraints does not affect any results one could calculate from the Hamiltonian. The naive free Hamiltonian resulting

from (4.2.17) is

$$H = \int_V \mathrm{d}^3x \ \sum_{\mathrm{i,j}} \left(\frac{1}{2} M_{\mathrm{ij}} \tilde{\psi}_{\mathrm{i}} \tilde{\psi}_{\mathrm{j}} + \frac{1}{2} G_{\mathrm{ij}} \nabla \tilde{\psi}_{\mathrm{i}} \cdot \nabla \tilde{\psi}_{\mathrm{j}} \right) + \frac{1}{2} \sum_{\mathrm{a,b}} \mathrm{g}_{\mathrm{ab}} \nabla \pi_{\mathrm{a}} \cdot \nabla \pi_{\mathrm{b}} \quad (4.2.24)$$

After defining Fourier components of fields as

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \exp\{\mathrm{i}\mathbf{k} \cdot \mathbf{x}\} \hat{\psi}(\mathbf{k}) \quad , \quad (4.2.25)$$

which satisfy the commutation relations

$$\{\hat{\psi}_{\mathrm{i}}(\mathbf{k}), \hat{P}_{\mathrm{j}}(\mathbf{k}')\} = \frac{1}{V} \delta_{\mathrm{i,j}} \delta_{\mathbf{k},-\mathbf{k}'} \quad , \quad (4.2.26)$$

we find the Hamiltonian in terms of these to be

$$H = V \sum_{\mathbf{k}} \sum_{\mathrm{i,j}} \left(\frac{1}{2} M_{\mathrm{ij}} \hat{\tilde{\psi}}_{\mathrm{i}}(\mathbf{k}) \hat{\tilde{\psi}}_{\mathrm{j}}(-\mathbf{k}) + \frac{1}{2} G_{\mathrm{ij}} |\mathbf{k}|^2 \hat{\tilde{\psi}}_{\mathrm{i}}(\mathbf{k}) \hat{\tilde{\psi}}_{\mathrm{j}}(-\mathbf{k}) \right) + \frac{1}{2} \sum_{\mathrm{a,b}} \mathrm{g}_{\mathrm{ab}} |\mathbf{k}|^2 \hat{\pi}_{\mathrm{a}}(\mathbf{k}) \hat{\pi}_{\mathrm{b}}(-\mathbf{k}) \quad . \quad (4.2.27)$$

After changing to the primed variables we can introduce creation and annihilation operators

$$a_{\mathrm{i}}^{\dagger}(\mathbf{k}) = \sqrt{\frac{V}{2}} \left[\hat{\psi}'_{\mathrm{i}}(-\mathbf{k}) - \mathrm{i} \hat{P}'_{\mathrm{i}}(-\mathbf{k}) \right], \quad a_{\mathrm{i}}(\mathbf{k}) = \sqrt{\frac{V}{2}} \left[\hat{\psi}'_{\mathrm{i}}(\mathbf{k}) + \mathrm{i} \hat{P}'_{\mathrm{i}}(\mathbf{k}) \right] \quad . \quad (4.2.28)$$

These satisfy the commutation relation

$$\{a_{\mathrm{i}}(\mathbf{k}), a_{\mathrm{j}}^{\dagger}(\mathbf{k}')\} = \delta_{\mathrm{i,j}} \delta_{\mathbf{k},\mathbf{k}'} \quad . \quad (4.2.29)$$

After replacing The canonical fields and momenta with these creation and annihilation operators, the Hamiltonian takes the form

$$\begin{aligned} H &= \sum_{i,j,\mathbf{k}} A_{\mathrm{i}}^{\dagger}(\mathbf{k}) [\mathcal{H}]_{\mathrm{i,j}} A_{\mathrm{j}}(\mathbf{k}) \quad , \\ A(\mathbf{k}) &= \left(a_1^{\dagger}(\mathbf{k}), a_1(-\mathbf{k}), a_2^{\dagger}(\mathbf{k}), a_2(-\mathbf{k}), a_3^{\dagger}(\mathbf{k}), a_3(-\mathbf{k}) \right) \quad . \end{aligned} \quad (4.2.30)$$

The quadratic form $[\mathcal{H}]$ can be diagonalized by a bosonic Bogoliubov transformation which preserves the commutation relations. The resulting spectrum consists

of three distinct dispersion relations which can be expressed using $\varepsilon(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2M}$ as

$$\omega_1(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})\left[\varepsilon(\mathbf{k}) + q)\right]} \quad ,$$

$$\omega_2(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})\left[\varepsilon(\mathbf{k}) + n(c_0 - c_1)\right] + 2c_1^2 n^2 - \frac{q^2}{2} - \sqrt{\begin{array}{l}\left(2c_1^2 n^2 - \frac{q^2}{2}\right)^2 - n(c_0 + 3c_1)(4c_1^2 n^2 - q^2)\varepsilon(\mathbf{k}) \\ + \left(n^2(c_0 + 3c_1)^2 - (c_0 + 2c_1)\frac{q^2}{c_1}\right)\varepsilon^2(\mathbf{k})\end{array}}} \quad ,$$

$$\omega_3(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})\left[\varepsilon(\mathbf{k}) + n(c_0 - c_1)\right] + 2c_1^2 n^2 - \frac{q^2}{2} + \sqrt{\begin{array}{l}\left(2c_1^2 n^2 - \frac{q^2}{2}\right)^2 - n(c_0 + 3c_1)(4c_1^2 n^2 - q^2)\varepsilon(\mathbf{k}) \\ + \left(n^2(c_0 + 3c_1)^2 - (c_0 + 2c_1)\frac{q^2}{c_1}\right)\varepsilon^2(\mathbf{k})\end{array}}} \quad . \tag{4.2.31}$$

In the limit $\mathbf{k} \to 0$ we find

$$\begin{aligned} \omega_1(0) &= 0 \quad , \\ \omega_2(0) &= \sqrt{\left(2n^2c_1^2 - \frac{q^2}{2}\right) - \left|2n^2c_1^2 - \frac{q^2}{2}\right|} \quad , \\ \omega_3(0) &= \sqrt{\left(2n^2c_1^2 - \frac{q^2}{2}\right) + \left|2n^2c_1^2 - \frac{q^2}{2}\right|} \quad . \end{aligned} \tag{4.2.32}$$

It is therefore clear that there will always be two gapless modes with linear dispersion relation and one gapped mode where the gap is

$$\Delta = \sqrt{\left|4n^2c_1^2 - q^2\right|} \quad . \tag{4.2.33}$$

Also in the case $c_1 = q = 0$ these dispersion relations reduce to (4.1.36) with $g = c_0$, as one would expect since the SO(2)xU(1) model reduces to the U(3) model.

These results are in agreement with results obtained from standard Bogoliubov-theory ([17]).

4.2.2 General Coset Construction in The Broken-Axisymmetry Phase

The general construction of a low energy Lagrangian based on the symmetry breaking pattern starts with the Lie algebra of the symmetry group SO(2)xU(1), which in this case is broken to the trivial subgroup. Following the notation of subsection 2.1 we therefore have $x_1 = \mathbf{X}_1$ and $x_2 = \mathbf{X}_2$ with all of the structure constants vanishing. This implies $\mathrm{D_{ab}} = \delta_{\mathrm{a,b}}$ and if we want to write down a quadratic Lagrangian density in terms of Goldstone fields only then it takes the form

$$\begin{aligned}\mathcal{L} &= \sum_{\mathrm{a,b}} \mathrm{v_a D_{ab}} \dot{\pi}_\mathrm{b} + \frac{1}{2} \sum_{\mathrm{a,b,c,d}} \mathrm{g}^t_{\mathrm{ab}} \mathrm{D_{ac} D_{bd}} \dot{\pi}_\mathrm{c} \dot{\pi}_\mathrm{d} - \frac{1}{2} \sum_{\mathrm{a,b,c,d}} \mathrm{g}^{\mathbf{x}}_{\mathrm{ab}} \mathrm{D_{ac} D_{bd}} \nabla \pi_\mathrm{c} \cdot \nabla \pi_\mathrm{d} \\ &= \mathrm{v}_1 \dot{\pi}_1 + \mathrm{v}_2 \dot{\pi}_2 + \frac{g^t_{11}}{2} \dot{\pi}_1^2 + \frac{g^t_{22}}{2} \dot{\pi}_2^2 + g^t_{12} \dot{\pi}_1 \dot{\pi}_2 - \frac{g^{\mathbf{x}}_{11}}{2} (\nabla \pi_1)^2 - \frac{g^{\mathbf{x}}_{22}}{2} (\nabla \pi_2)^2 - g^{\mathbf{x}}_{12} \nabla \pi_1 \cdot \pi_2\end{aligned} \tag{4.2.34}$$

The first two terms can be dropped, being total derivatives. For the canonical momenta one finds

$$P_1 = g^t_{11} \dot{\pi}_1 + g^t_{12} \dot{\pi}_2 \quad , \quad P_2 = g^t_{22} \dot{\pi}_2 + g^t_{12} \dot{\pi}_1 \tag{4.2.35}$$

Unless $g^t_{11} g^t_{22} = {g^t_{12}}^2$, these relations are invertible and there are no constraints. The justification for assuming this case is that it yields the correct number of gapless modes. One therefore has to have some prior knowledge of the system or additional arguments in order to choose between different possibilities when following the general scheme. Inverting (4.2.35) gives

$$\begin{aligned}\dot{\pi}_1 &= \frac{g^t_{22} P_1 - g^t_{12} P_2}{g^t_{11} g^t_{22} - {g^t_{12}}^2} \quad , \quad \dot{\pi}_2 = \frac{g^t_{11} P_2 - g^t_{12} P_1}{g^t_{11} g^t_{22} - {g^t_{12}}^2} \\ \mathcal{H} &= \frac{1}{2(g^t_{11} g^t_{22} - {g^t_{12}}^2)} \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} g^t_{22} & -g^t_{12} \\ -g^t_{12} & g^t_{11} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \nabla \pi_1 & \nabla \pi_2 \end{bmatrix} \begin{bmatrix} g^{\mathbf{x}}_{11} & g^{\mathbf{x}}_{12} \\ g^{\mathbf{x}}_{12} & g^{\mathbf{x}}_{22} \end{bmatrix} \begin{bmatrix} \nabla \pi_1 \\ \nabla \pi_2 \end{bmatrix}\end{aligned} \tag{4.2.36}$$

The Hamiltonian density as a function of fields and momenta is therefore a block diagonal quadratic form and by redefining the fields and momenta using an O(2) rotation and its inverse we can diagonalize either one of the blocks while preserving the commutation relations.Thus, without loss of generality, we can take the matrix $[g^t]$ to be diagonal and therefore the Hamiltonian density in terms of Fourier components to be

$$\mathcal{H} = \frac{1}{2} \begin{bmatrix} \hat{P}_1(\mathbf{k}) & \hat{P}_2(\mathbf{k}) \end{bmatrix} \begin{bmatrix} 1/g^t_{22} & 0 \\ 0 & 1/g^t_{11} \end{bmatrix} \begin{bmatrix} \hat{P}_1(-\mathbf{k}) \\ \hat{P}_2(-\mathbf{k}) \end{bmatrix} + \frac{1}{2} |\mathbf{k}|^2 \begin{bmatrix} \hat{\pi}_1(\mathbf{k}) & \hat{\pi}_2(\mathbf{k}) \end{bmatrix} \begin{bmatrix} g^{\mathbf{x}}_{11} & g^{\mathbf{x}}_{12} \\ g^{\mathbf{x}}_{12} & g^{\mathbf{x}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\pi}_1(-\mathbf{k}) \\ \hat{\pi}_2(-\mathbf{k}) \end{bmatrix} \tag{4.2.37}$$

Defining creation and annihilation operators

$$a_{\mathrm{i}}^{\dagger}(\mathbf{k}) = \sqrt{\frac{V}{2}}\left[\hat{\pi}_{\mathrm{i}}(-\mathbf{k}) - \mathrm{i}\hat{P}_{\mathrm{i}}(-\mathbf{k})\right], \quad a_{\mathrm{i}}(\mathbf{k}) = \sqrt{\frac{V}{2}}\left[\hat{\pi}_{\mathrm{i}}(\mathbf{k}) + \mathrm{i}\hat{P}_{\mathrm{i}}(\mathbf{k})\right] \tag{4.2.38}$$

turns the Hamiltonian into the quadratic form

$$\mathcal{H} = \frac{1}{4}\begin{bmatrix} a_1(\mathbf{k}) \\ a_1^{\dagger}(-\mathbf{k}) \\ a_2(\mathbf{k}) \\ a_2^{\dagger}(-\mathbf{k}) \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{g_{11}^t}+g_{11}^{\mathbf{x}}|\mathbf{k}|^2 & -\frac{1}{g_{11}^t}+g_{11}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 \\ -\frac{1}{g_{11}^t}+g_{11}^{\mathbf{x}}|\mathbf{k}|^2 & \frac{1}{g_{11}^t}+g_{11}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 \\ g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & \frac{1}{g_{22}^t}+g_{22}^{\mathbf{x}}|\mathbf{k}|^2 & -\frac{1}{g_{22}^t}+g_{22}^{\mathbf{x}}|\mathbf{k}|^2 \\ g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & g_{21}^{\mathbf{x}}|\mathbf{k}|^2 & -\frac{1}{g_{22}^t}+g_{22}^{\mathbf{x}}|\mathbf{k}|^2 & \frac{1}{g_{22}^t}+g_{22}^{\mathbf{x}}|\mathbf{k}|^2 \end{bmatrix} \cdot \begin{bmatrix} a_1^{\dagger}(\mathbf{k}) \\ a_1(-\mathbf{k}) \\ a_2^{\dagger}(\mathbf{k}) \\ a_2(-\mathbf{k}) \end{bmatrix}, \tag{4.2.39}$$

which can be diagonalized by a bosonic Bogoliubov transformation:

$$\begin{aligned} H_0 = \sum_{\mathbf{k}} & \sqrt{\frac{g_{22}^t g_{11}^{\mathbf{x}} + g_{11}^t g_{22}^{\mathbf{x}} + \sqrt{{g_{22}^t}^2 {g_{11}^{\mathbf{x}}}^2 + {g_{11}^t}^2 {g_{22}^{\mathbf{x}}}^2 + 2 g_{11}^t g_{22}^t \left(2{g_{21}^{\mathbf{x}}}^2 - g_{11}^{\mathbf{x}} g_{22}^{\mathbf{x}}\right)}}{2 g_{11}^t g_{22}^t}} |\mathbf{k}| b_1^{\dagger}(\mathbf{k}) b_1(\mathbf{k}) \\ & + \sqrt{\frac{g_{22}^t g_{11}^{\mathbf{x}} + g_{11}^t g_{22}^{\mathbf{x}} - \sqrt{{g_{22}^t}^2 {g_{11}^{\mathbf{x}}}^2 + {g_{11}^t}^2 {g_{22}^{\mathbf{x}}}^2 + 2 g_{11}^t g_{22}^t \left(2{g_{21}^{\mathbf{x}}}^2 - g_{11}^{\mathbf{x}} g_{22}^{\mathbf{x}}\right)}}{2 g_{11}^t g_{22}^t}} |\mathbf{k}| b_2^{\dagger}(\mathbf{k}) b_2(\mathbf{k}) \end{aligned} \tag{4.2.40}$$

The general construction therefore predicts that in the low energy limit there should be two relevant modes with linear dispersions and different sound velocities.

4.2.3 Interactions

Unlike the U(3)-symmetric Lagrangian there are no interaction terms which originate from the lowest order derivative terms in the general Lagrangian which provides the free part of the Lagrangian. This is simply due to the fact that $\mathrm{D}_{\mathrm{ab}} = \delta_{\mathrm{a,b}}$. Therefore all of the allowed interactions must be purely products of derivatives of the Goldstone-fields. The great disadvantage in this case is that their coefficients do not follow by simply matching those of the free Lagrangian to the corresponding ones appearing in the explicitly reparameterized version. This showcases that the power of the general coset construction depends on the structure of the symmetry breaking pattern. It is most useful for systems where the number of broken generators is low, which implies a small number of Goldstone-fields, while having a large unbroken group H, which provides great constraints on the coefficients appearing in the Lagrangian. Additionally, if the Lie algebra of the symmetry generators is non-abelian, then in general the coefficients of the free Lagrangian are not independent of the ones for the most relevant interaction terms, such that one can obtain both the spectrum and the interactions simultaneously.

For our Spin-1 system we need to do some more work in order to see which of

the allowed interaction terms are in fact present.

4.2.4 Matching Coefficients

In order to get the interactions we can use the reparameterized original Lagrangian within the path integral formulation and integrate out the massive fields, which are absent in the general construction, since we are only interested in the effective theory for the Goldstone fields and corresponding modes:

$$
\begin{aligned}
&\int \mathcal{D}(\pi)\mathcal{D}(\tilde{\psi}) \exp\left[\int_x \mathcal{L}'(\pi,\tilde{\psi})\right] \\
&= \int \mathcal{D}(\hat{\pi}) \left\{ \exp\left[2\pi V \int_\omega \sum_{\mathbf{k}} -\frac{1}{2} \sum_{a,b=1}^{2} g_{ab} |\mathbf{k}|^2 (\hat{\pi}_a)_{\mathbf{k},\omega} (\hat{\pi}_b)_{-\mathbf{k},-\omega} \right] \right. \\
&\quad \cdot \int \mathcal{D}(\hat{\tilde{\psi}}) \exp\left[2\pi V \int_{\omega,\omega'} \sum_{\mathbf{k},\mathbf{k}'} -\frac{1}{2} \sum_{i,j=1}^{4} (\hat{\tilde{\psi}}_i)_{\mathbf{k},\omega} \left[\left(M_{ij} + |\mathbf{k}|^2 G_{ij} + i\omega T_{ij} \right) \delta_{\mathbf{k},-\mathbf{k}'} \delta(\omega+\omega') \right.\right. \\
&\qquad\qquad \left. + \sum_{a=1}^{2} Q_{aij}\, i\left(-\omega-\omega'\right) (\hat{\pi}_a)_{-\mathbf{k}-\mathbf{k}',-\omega-\omega'} + \mathcal{O}(|\mathbf{k}|^2\pi^2) \right] (\hat{\tilde{\psi}}_j)_{\mathbf{k}',\omega'} \\
&\qquad\qquad \left. \left. + \sum_{a=1}^{2} \sum_{i=1}^{4} N_{ai} (\hat{\pi}_a)_{\mathbf{k},\omega} (i\omega) \delta_{\mathbf{k},-\mathbf{k}} \delta(\omega+\omega') (\hat{\tilde{\psi}}_j)_{\mathbf{k}',\omega'} \right] \right\} \\
&\propto \int \mathcal{D}(\hat{\pi}) \exp\left[2\pi V \int_\omega \sum_{\mathbf{k}} -\frac{1}{2} \sum_{a,b=1}^{2} g_{ab} |\mathbf{k}|^2 (\hat{\pi}_a)_{\mathbf{k},\omega} (\hat{\pi}_b)_{-\mathbf{k},-\omega} \right. \\
&\qquad\qquad \left. + \frac{1}{2} \sum_{a,b=1}^{2} \sum_{i,j=1}^{4} \omega^2 \left(M_{ij} + |\mathbf{k}|^2 G_{ij} + i\omega T_{ij} \right)^{-1} N_{ai} N_{bj} (\hat{\pi}_a)_{\mathbf{k},\omega} (\hat{\pi}_b)_{-\mathbf{k},-\omega} \right] \\
&= \int \mathcal{D}(\hat{\pi}) \exp\left[2\pi V \int_\omega \sum_{\mathbf{k}} -\frac{1}{2} \sum_{a,b=1}^{2} g_{ab} |\mathbf{k}|^2 (\hat{\pi}_a)_{\mathbf{k},\omega} (\hat{\pi}_b)_{-\mathbf{k},-\omega} \right. \\
&\qquad\qquad \left. + \frac{1}{2}\omega^2 \left(\frac{1+2Q}{4c_1 Q} (\hat{\pi}_1)_{\mathbf{k},\omega} (\hat{\pi}_1)_{-\mathbf{k},-\omega} + \frac{1}{c_0+c_1} (\hat{\pi}_2)_{\mathbf{k},\omega} (\hat{\pi}_2)_{-\mathbf{k},-\omega} \right) + \dots \right] \\
&= \int \mathcal{D}(\hat{\pi}) \exp\left[\int_x -\frac{1}{2} \sum_{a,b=1}^{2} g_{ab} \nabla\pi_a \cdot \nabla\pi_a + \frac{1}{2} \left(\frac{1+2Q}{4c_1 Q} \dot{\pi}_1^2 + \frac{1}{c_0+c_1} \dot{\pi}_2^2 \right) + \dots \right]
\end{aligned}
\tag{4.2.41}
$$

The integration over the massive fields is again done by completing the square. The third equality holds up to terms of third order in the fields and in the penultimate step the inverse quadratic form has been expanded to zeroth order in $|\mathbf{k}|$ and ω to get the lowest order terms contributing to the free action. By comparing (4.2.17) and (4.2.34) we can match the coefficients:

$$
g_{11}^t = \frac{1+2Q}{4c_1 Q}, \quad g_{22}^t = \frac{1}{c_0+c_1}, \quad g_{11}^{\mathbf{x}} = \frac{n(1+2Q)}{2M}, \quad g_{22}^{\mathbf{x}} = \frac{n}{M}, \quad g_{12}^{\mathbf{x}} = g_{21}^{\mathbf{x}} = 0
\tag{4.2.42}
$$

By plugging these values into (4.2.40) we find the dispersion relations

$$\omega_1(\mathbf{k}) = \sqrt{\frac{q}{2M}}|\mathbf{k}| \quad , \quad \omega_2(\mathbf{k}) = \sqrt{\frac{n(c_0+c_1)}{M}}|\mathbf{k}| \qquad , \tag{4.2.43}$$

which are in accordance with the low energy limit of (4.2.31) if we only take the ungapped modes into account. In order to obtain the interaction terms we need to take into account the term in the propagator for ψ which is proportional to π. The inverse of this propagator to first order in π and to lowest order in derivatives is given by

$$\begin{aligned}
&\left[\left(M_{ij} + |\mathbf{k}|^2 G_{ij} + i\omega T_{ij}\right)\delta_{\mathbf{k},-\mathbf{k}'}\delta(\omega+\omega') + \sum_{a=1}^{2} Q_{aij}\, i\left(-\omega-\omega'\right)\left(\hat{\pi}_a\right)_{-\mathbf{k}-\mathbf{k}',-\omega-\omega'}\right]^{-1} \\
\approx &(M)^{-1}_{ij}\delta_{\mathbf{k},-\mathbf{k}'}\delta(\omega+\omega') + \sum_{a=1}^{2}\sum_{n,m=1}^{4}(M)^{-1}_{in}Q_{anm}(M)^{-1}_{mj}\, i\left(-\omega-\omega'\right)\left(\hat{\pi}_a\right)_{-\mathbf{k}-\mathbf{k}',-\omega-\omega'}
\end{aligned} \tag{4.2.44}$$

The second term gives rise to interaction terms $\propto \dot{\pi}^3$. Both the propagator and the term linear in ψ also contain a term $\propto |\mathbf{k}|^2\pi^2$ which neither contributes to the free action nor to the third order interaction terms. In particular there is no term $\propto \dot{\pi}(\nabla\pi)^2$ which would contribute interactions in terms of the quasiparticle creation and annihilation operators that scale as $|\mathbf{k}|^{3/2}$. The present interaction terms $\propto \dot{\pi}^3$ scale as $|\mathbf{k}|^{3/2}$ as well.

5 Scaling Analysis

5.1 Spin-1 - Scaling of Quasi-Particle Occupation Number

5.1.1 Spatio-Temporal Scaling

For the purpose of studying the dynamics near non-thermal fixed points of the Spin-1 bose gas, we can perform a scaling analysis of the scattering integrals which can be derived from the low energy effective Hamiltonian and by assuming that kinetic theory is a valid approximation. For reference see e.g. [2],[4],[12],[14]. The statistical function for quasi-particle excitations in terms of the creation and annihilation operators introduced in (4.2.40) reads

$$f_{\mathrm{a}}(\mathbf{k},t) \coloneqq \left\langle b_{\mathrm{a}}^{\dagger}(\mathbf{k},t) b_{\mathrm{a}}(\mathbf{k},t) \right\rangle \tag{5.1.1}$$

Constructing the coherent state path integral for a Schwinger-Keldish contour from the diagonalized Hamiltonian gives us

$$\begin{aligned} Z = \int \mathcal{D}(b)\mathcal{D}(\bar{b}) \exp\Bigg[\int_{t,\mathcal{C}} \sum_{\mathbf{k}} \sum_{\mathrm{a}} & \mathrm{i}\bar{b}_{\mathrm{a}}(\mathbf{k})\partial_t b_{\mathrm{a}}(\mathbf{k}) - \omega_{\mathrm{a}}(\mathbf{k})\bar{b}_{\mathrm{a}}(\mathbf{k}) b_{\mathrm{a}}(\mathbf{k}) \\ & - \mathcal{O}\left(|\mathbf{k}|^{\frac{3}{2}} (b/\bar{b})^3 \right) - \mathcal{O}\left(|\mathbf{k}|^2 (b/\bar{b})^4 \right) - \dots \Bigg] \end{aligned} \tag{5.1.2}$$

Where $(b/\bar{b})^{\mathrm{n}}$ denotes terms which are of third total order in b and $\bar{b}$. The free propagator is therefore

$$\mathcal{G}^{(0)}(\omega,\mathbf{k})_{\mathrm{ab}} \propto \frac{\delta_{\mathrm{a,b}}}{\omega - \omega_{\mathrm{a}}(\mathbf{k}) + i0^+} \quad , \tag{5.1.3}$$

which in the diagrammatic calculations of the self energy contributes Dirac-distributions $\delta(\omega(\mathbf{k_1}) \pm \omega(\mathbf{k_2}) \pm \dots)$ in the dispersions appearing in the integrals. By combining the 3- and 4-vertices Fig. 5.1 of the lowest order interactions we obtain the set of diagrams Fig.5.2 which contribute to the proper self-energy and where internal lines are now to be taken as full propagators. Solid lines represent the mode associ-

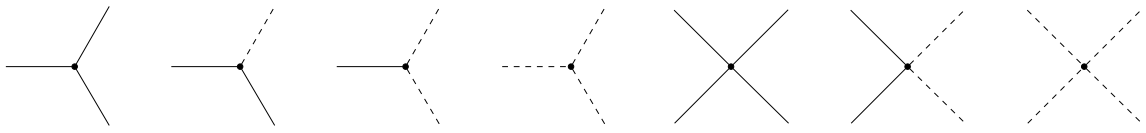

Figure 5.1: 3- and 4- vertices appearing in the interation Hamiltonian

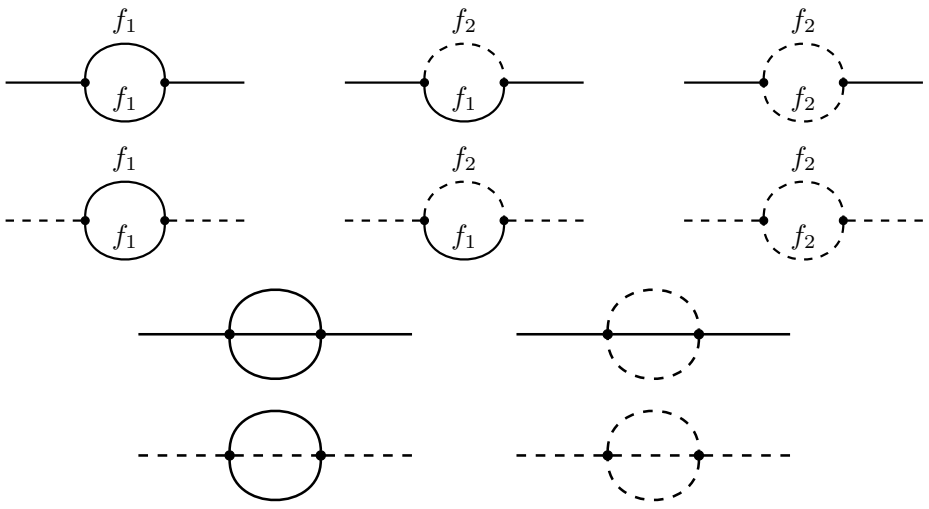

Figure 5.2: Lowest order diagrams contributing to the proper self energy

ated with π_1 which originates from the magnetic field breaking the SO(3) symmetry, while dashed lines represent the Bogoliubov-mode associated with total-phase transformations. Within the framework of kinetic theory where all processes are assumed to involve only on-shell contributions this leads to the set of kinetic equations

$$\partial_t f_{\mathrm{i}}(\mathbf{k}) = I_{\mathrm{i},3}[f](\mathbf{k},t) + I_{\mathrm{i},4}[f](\mathbf{k},t) \qquad , \ \mathrm{i} = 1,2. \tag{5.1.4}$$

The scattering integrals receive contributions from all of the self-energy diagrams, i.e.

$$\begin{aligned}
I_{1,3}[f](\mathbf{k},t) &\propto \int_{\mathbf{q}} |\mathbf{k}/\mathbf{q}|^3 \left(\lambda_{1,1} f_1^2(\mathbf{k}/\mathbf{q},t) + \lambda_{1,2} f_1 f_2(\mathbf{k}/\mathbf{q},t) + \lambda_{1,3} f_2^2(\mathbf{k}/\mathbf{q},t)\right) \ \delta(\omega(\mathbf{k}) \pm \omega(\mathbf{q}) \pm \dots), \\
I_{2,3}[f](\mathbf{k},t) &\propto \int_{\mathbf{q}} |\mathbf{k}/\mathbf{q}|^3 \left(\lambda_{2,1} f_1^2(\mathbf{k}/\mathbf{q},t) + \lambda_{2,2} f_1 f_2(\mathbf{k}/\mathbf{q},t) + \lambda_{2,3} f_2^2(\mathbf{k}/\mathbf{q},t)\right) \ \delta(\omega(\mathbf{k}) \pm \omega(\mathbf{q}) \pm \dots), \\
I_{1,4}[f](\mathbf{k},t) &\propto \int_{\mathbf{q},\mathbf{p}} |\mathbf{k}/\mathbf{q}/\mathbf{p}|^4 \left(\gamma_{1,1} f_1^3(\mathbf{k}/\mathbf{q}/\mathbf{p},t) + \gamma_{1,2} f_1 f_2^2(\mathbf{k}/\mathbf{q}/\mathbf{p},t)\right) \ \delta(\omega(\mathbf{k}) \pm \omega(\mathbf{q}) \pm \omega(\mathbf{p}) \pm \dots), \\
I_{2,4}[f](\mathbf{k},t) &\propto \int_{\mathbf{q},\mathbf{p}} |\mathbf{k}/\mathbf{q}/\mathbf{p}|^4 \left(\gamma_{2,1} f_1^3(\mathbf{k}/\mathbf{q}/\mathbf{p},t) + \gamma_{2,2} f_1 f_2^2(\mathbf{k}/\mathbf{q}/\mathbf{p},t)\right) \ \delta(\omega(\mathbf{k}) \pm \omega(\mathbf{q}) \pm \omega(\mathbf{p}) \pm \dots),
\end{aligned} \tag{5.1.5}$$

where the notation is supposed to indicate that there are several terms contributing in each case which differ in the way the momenta are combined. Since the structure is the same for all of these and the scaling behaviour follows from this structure alone, we do not write them down individually. Keeping only the highest powers in f_{i} will later be justified by looking at the momentum dependence of f. The scaling ansatz for self-similar time evolution is given by

$$f_{\mathrm{i}}(\mathbf{k},t) = s^{\frac{\alpha_{\mathrm{i}}}{\beta}} f(s\mathbf{k}, s^{-\frac{1}{\beta}} t) \quad . \tag{5.1.6}$$

We allow for a different scaling exponent α_i for the different modes but assume that they both obey self-similar time evolution as well as β being the same since this required in order to obtain a well defined scaling exponent for scattering integrals. Consistency with the time evolution equation allows one to determine the exponents $\alpha_\mathrm{i}, \beta_\mathrm{i}$. For that purpose we need to calculate the scaling coefficients of the scattering integrals. Since each integral involves a sum over terms of different order in the statistical functions f_i, one can in general not identify a single, overall scaling exponent. We will therefore make the ansatz of assuming either $f_1 \gg f_2$ or $f_2 \gg f_1$. In this case the terms of highest power in the corresponding statistical function will dictate the scaling of each scattering integral. Under the assumption $f_1 \gg f_2$ and with d denoting the dimensionality of space we find

$$\begin{aligned}
I_{1,3}[f](\mathbf{k},t) &= s^{-\mu_{1,3}} I_{1,3}[f](s\mathbf{k}, s^{-\frac{1}{\beta}}t) \qquad \text{with} \quad \mu_{1,3} = d+2-2\frac{\alpha_1}{\beta} \\
I_{1,4}[f](\mathbf{k},t) &= s^{-\mu_{1,4}} I_{1,4}[f](s\mathbf{k}, s^{-\frac{1}{\beta}}t) \qquad \text{with} \quad \mu_{1,4} = 2d+3-3\frac{\alpha_1}{\beta} \\
I_{2,3}[f](\mathbf{k},t) &= s^{-\mu_{2,3}} I_{2,3}[f](s\mathbf{k}, s^{-\frac{1}{\beta}}t) \qquad \text{with} \quad \mu_{2,3} = d+2-2\frac{\alpha_1}{\beta} \\
I_{2,4}[f](\mathbf{k},t) &= s^{-\mu_{2,4}} I_{2,4}[f](s\mathbf{k}, s^{-\frac{1}{\beta}}t) \qquad \text{with} \quad \mu_{1,4} = 2d+3-3\frac{\alpha_1}{\beta}.
\end{aligned} \tag{5.1.7}$$

The kinetic equations dictate that $\alpha_\mathrm{i} = 1 - \mu_\mathrm{i}\beta$ where μ_i is to be taken either as $\mu_{\mathrm{i,l}}$ if the scattering integral $I_{\mathrm{i,l}}$ dominates over the other one or as both $\mu_\mathrm{i} = \mu_{\mathrm{i},3} = \mu_{\mathrm{i},4}$ if neither one is to be neglected. In the first case one already obtains a complete set of equations that determined the exponents uniquely while in the second case another condition needs to be added. One possibility is total energy conservation in which case the relation

$$\begin{aligned}
E &= \int_\mathbf{k} \omega_1(\mathbf{k}) f_1(\mathbf{k},t) + \omega_2(\mathbf{k}) f_2(\mathbf{k},t) \\
&= \int_\mathbf{k} s^d \left(\omega_1(s\mathbf{k}) f_1(s\mathbf{k}, s^{-\frac{1}{\beta}}t) + \omega_2(s\mathbf{k}) f_2(s\mathbf{k}, s^{-\frac{1}{\beta}}t) \right) \\
&= \int_\mathbf{k} s^d \left(s^{1-\frac{\alpha_1}{\beta}} \omega_1(\mathbf{k}) f_1(\mathbf{k},t) + s^{1-\frac{\alpha_2}{\beta}} \omega_2(\mathbf{k}) f_2(\mathbf{k},t) \right)
\end{aligned} \tag{5.1.8}$$

requires $\alpha_1 = \alpha_2 = \beta(d+1)$. Alternatively one oftem seems to find that quasi-particle numbers are conserved in certain regimes and so we can make the ansatz that each particle number is conserved and the relation

$$n_\mathrm{i} = \int_\mathbf{k} f_\mathrm{i}(\mathbf{k},t) = \int_\mathbf{k} s^d f_\mathrm{i}(s\mathbf{k}, s^{-\frac{1}{\beta}}t) = \int_\mathbf{k} s^d s^{-\frac{\alpha_\mathrm{i}}{\beta}} f_\mathrm{i}(\mathbf{k},t) \tag{5.1.9}$$

provides the conditions $\alpha_\mathrm{i} = \beta d$, which again requires that $\alpha_1 = \alpha_2$. We can now go through all of the possibilities starting with the consistent solution obtained by

imposing equal scaling exponents for 3- and 4- scattering integrals:

$$\mu_{\mathrm{i},3} = \mu_{\mathrm{i},4} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 1 + d \quad , \quad \beta = 1 \tag{5.1.10}$$

In all of the other cases we need to separately consider the possibility of the I_l integrals being dominant over the other one and vice versa. For the case of number conservation one finds

$$\begin{aligned} l = 3: \qquad & \alpha_1 = \alpha_2 = \frac{d}{2} \quad , \quad \beta = \frac{1}{2} \\ l = 4: \qquad & \alpha_1 = \alpha_2 = \frac{d}{3} \quad , \quad \beta = \frac{1}{3} \end{aligned} \tag{5.1.11}$$

For the case of energy conservation, where we will use primed exponents, one finds

$$\begin{aligned} l = 3: \qquad & \alpha_1' = \alpha_2' = d + 1 \quad , \quad \beta' = 1 \\ l = 4: \qquad & \alpha_1' = \alpha_2' = d + 1 \quad , \quad \beta' = 1 \end{aligned} \tag{5.1.12}$$

Of course the result obtained by assuming a certain scattering integral dominating over the other should always be consistent with that assumption. After inserting the obtained solutions for α and β into the equations for the scattering exponents (5.1.7) we identify the smaller exponent as the one whose corresponding integral is the dominant one at sufficiently late times and small momenta. That this should be the case is indicated by their scaling relations in (5.1.7).

For the case of number conservation we find

$$\begin{aligned} I_3 \text{ dominant} \Rightarrow \qquad & \mu_{\mathrm{i},3} = 2 - d \quad , \quad \mu_{\mathrm{i},4} = 3 - d \\ I_4 \text{ dominant} \Rightarrow \qquad & \mu_{\mathrm{i},3} = 2 - d \quad , \quad \mu_{\mathrm{i},4} = 3 - d \quad . \end{aligned} \tag{5.1.13}$$

Since in both cases $\mu_{\mathrm{i},3} < \mu_{\mathrm{i},4}$ we conclude that that only the assumption of I_3 dominating leads to a consistent solution.

For the case of energy conservation we find

$$\begin{aligned} I_3 \text{ dominant} \Rightarrow \qquad & \mu_{\mathrm{i},3}' = -d \quad , \quad \mu_{\mathrm{i},4} = -d \\ I_4 \text{ dominant} \Rightarrow \qquad & \mu_{\mathrm{i},3}' = -d \quad , \quad \mu_{\mathrm{i},4} = -d \quad , \end{aligned} \tag{5.1.14}$$

meaning that the relative importance is expected to not change as time goes on. However, this poses no inconsistency, since there is a common scaling exponent for both integrals. This solution is also the same as the one obtained by making the ansatz that there should be a common exponent in the first place.

The results one obtains after assuming that $f_2 \gg f_1$ are exactly identical and since in all cases the scaling exponents are the same for f_1 and f_2, it seems plausible that one can drop the assumption of either one f_i dominating and extrapolate these results by choosing $\alpha_1 = \alpha_2$ from the beginning. From now on we will make this

more special ansatz for all calculations.

5.1.2 Shape of the Scaling Function

The scaling ansatz can be rewritten as

$$f(\mathbf{k},t) = (t/t_{\mathrm{ref}})^{\alpha} f_{\mathrm{S}}([t/t_{\mathrm{ref}}]^{\beta}\mathbf{k} \quad , \tag{5.1.15}$$

with the universal scaling function

$$f_{\mathrm{S}}(\mathbf{k}) = f(\mathbf{k}, t_{\mathrm{ref}}) \quad . \tag{5.1.16}$$

If one makes a similar scaling ansatz for the spatial form of the scaling function itself, i.e.

$$f_{\mathrm{S}}(\mathbf{k}) = s^{\kappa} f_{\mathrm{S}}(s\mathbf{k}) \tag{5.1.17}$$

then the kinetic equation yields

$$(\alpha - \beta) f_{\mathrm{S}}(\mathbf{k}) = t_{\mathrm{ref}} I[f_{\mathrm{S}}](\mathbf{k}) \quad . \tag{5.1.18}$$

The scaling behaviour of f_{S} implies that the scattering integral obeys fixed time scaling as well, namely

$$I_l[f_{\mathrm{S}}](\mathbf{k}) = s^{-\mu_{\kappa,l}} I_l[f_{\mathrm{S}}](s\mathbf{k}, t_{\mathrm{ref}}) \quad , \tag{5.1.19}$$

assuming that I_l dominates the fixed time kinetic equation. We therefore find

$$\kappa = -\mu_{\kappa,l} \quad . \tag{5.1.20}$$

The spatial scaling exponents of the scattering integrals follow straightforwardly from (5.1.5):

$$\begin{aligned} l = 3: &\qquad \mu_{\kappa,3} = d + 2 - 2\kappa \\ l = 4: &\qquad \mu_{\kappa,4} = 2d + 3 - 3\kappa \end{aligned} \tag{5.1.21}$$

This results in the following scaling exponents:

$$\begin{aligned} l = 3: &\qquad \kappa_3 = d + 2 \\ l = 4: &\qquad \kappa_4 = d + \frac{3}{2} \end{aligned} \tag{5.1.22}$$

Since $\kappa > 0$ in all cases, we can assume that for small momenta $f(\mathbf{k},t) \gg 1$ and

hence only the highest power in f needs to be kept in the scattering integral. Inserting the same $\kappa = \kappa_l$ into (5.1.21) gives

$$\mu_{\kappa,3} - \mu_{\kappa,4} = \kappa_l - d - 1 > 0 \tag{5.1.23}$$

independently of l. The scaling of the scattering integrals indicates that at small momenta that I_l which has the smallest $\mu_{\kappa,l}$ dominates and so for consistency we must have $\kappa = \kappa_4$.

5.2 Spin-1 - Scaling of Goldstone-Field Correlator

5.2.1 Spatio-Temporal Scaling

Instead of the quasi-particle correlation function we can also look at the correlator of the Goldstone-fields:

$$F_{\mathrm{ab}}(\mathbf{k}, t) := \langle \pi_{\mathrm{a}}(\mathbf{k}, t)\pi_{\mathrm{b}}(-\mathbf{k}, t)\rangle \tag{5.2.1}$$

Since the quadratic part of the Lagrangian is of the form

$$\mathcal{L}_{\mathrm{free}} = \frac{1}{2}\pi_{\mathrm{a}}(\mathbf{k}, \omega)\left[\omega^2\sigma_{\mathrm{ab}} - g^{\mathbf{x}}_{\mathrm{ab}}|\mathbf{k}|^2\right]\pi_{\mathrm{b}}(-\mathbf{k}, -\omega) \tag{5.2.2}$$

we conclude that $\mathcal{G}^{(0)} \propto \frac{1}{(\omega+i0^+)^2-\omega(\mathbf{k})^2}$ and therefore the spectral function takes in the on-shell approximation the form

$$\rho(\mathbf{k}, \omega) \propto \frac{1}{\omega(\mathbf{k})}\left[\delta(\omega - \omega(\mathbf{k})) - \delta(\omega + \omega(\mathbf{k}))\right] \tag{5.2.3}$$

The interaction vertices contribute the following scaling factors if $\mathbf{k} \to s\mathbf{k}$ where $\omega(\mathbf{k}) \propto |\mathbf{k}|$ is used:

$$\begin{aligned} \dot{\pi}^3 &\to \omega(\mathbf{k})\omega(\mathbf{p})\omega(\mathbf{q}) && \propto |\mathbf{k}||\mathbf{p}||\mathbf{q}| \\ (\nabla\pi)^2\dot{\pi} &\to \mathbf{k}\cdot\mathbf{p}\,\omega(\mathbf{q}) && \propto |\mathbf{k}||\mathbf{p}||\mathbf{q}| \end{aligned} \tag{5.2.4}$$

Therefore the 3-vertex $\gamma(\mathbf{k}, \mathbf{p}, \mathbf{q})$ scales as

$$\gamma(s\mathbf{k}, s\mathbf{p}, s\mathbf{q}) = s^3\gamma(\mathbf{k}, \mathbf{p}, \mathbf{q}) \quad . \tag{5.2.5}$$

Similarly we have for the 4-vertex:

$$\begin{aligned} \dot{\pi}^4 &\to \omega(\mathbf{k})\omega(\mathbf{p})\omega(\mathbf{q})\omega(\mathbf{r}) && \propto |\mathbf{k}||\mathbf{p}||\mathbf{q}||\mathbf{r}| \\ (\nabla\pi)^2\dot{\pi}^2 &\to \mathbf{k}\cdot\mathbf{p}\,\omega(\mathbf{q})\omega(\mathbf{r}) && \propto |\mathbf{k}||\mathbf{p}||\mathbf{q}||\mathbf{r}| \\ (\nabla\pi)^4 &\to (\mathbf{k}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{r}) + \text{perm.} && \propto |\mathbf{k}||\mathbf{p}||\mathbf{q}||\mathbf{r}| \end{aligned} \tag{5.2.6}$$

Which implies that the 4-vertex $\lambda(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{r})$ scales as

$$\lambda(s\mathbf{k},s\mathbf{p},s\mathbf{q},s\mathbf{r}) = s^4\lambda(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{r}) \quad . \tag{5.2.7}$$

The scattering integrals and their scaling are then given by

$$\begin{aligned} I_3(\mathbf{k},t) &\propto \int_{\mathbf{p}} |\gamma|^2 \frac{1}{\omega(\mathbf{k}/\mathbf{p})^3}\, F^2\, \delta(\omega(\mathbf{k}) \pm \dots) && \to s^{d+6-3-2\frac{\alpha}{\beta}-1}\; I_3(\mathbf{k},t) \\ I_4(\mathbf{k},t) &\propto \int_{\mathbf{p},\mathbf{q}} |\lambda|^2 \frac{1}{\omega(\mathbf{k}/\mathbf{p}/\mathbf{q})^4}\, F^3\, \delta(\omega(\mathbf{k}) \pm \dots) && \to s^{2d+8-4-3\frac{\alpha}{\beta}-1}\; I_4(\mathbf{k},t) \end{aligned} \tag{5.2.8}$$

Unsurprisingly, one obtains the same scaling exponents as before and the solutions for α and β are the ones calculated above.

5.2.2 Relating Goldstone-Fields to Experimental Observables

In experiments with Spin-1 bose gases the universal scaling dynamics has already been observed and the data seems to indicate that essentially only one degree of freedom needs to be taken into account [13]. Using the complex spin matrices $\vec{\mathbf{F}}$ and the creation and annihilation operators $\psi^\dagger_{\mathrm{m}}, \psi_{\mathrm{m}}$ for the magnetic sublevel $\mathrm{m} \in \{1, 0, -1\}$ one can define the spin fluctuation operators

$$\hat{F}_{\mathrm{i}} = \sum_{\mathrm{m},\mathrm{m}'} \psi^\dagger_{\mathrm{m}} [\mathbf{F}_{\mathrm{i}}]_{\mathrm{m}\mathrm{m}'} \psi_{\mathrm{m}'}. \tag{5.2.9}$$

In the experiment the components in the x-y-plane are measured and it is found that the absolute value of this transverse spin is essentially fixed while it phase is the dynamic degree of freedom. By using the parameterization in terms of Goldstone-fields and massive fields that we have adopted in this work, one can identify this phase as one of the Goldstone-fields. By neglecting fluctuations in all massive fields we have

$$\psi_1 = \mathrm{e}^{\mathrm{i}(\pi_1+\pi_2)}\sqrt{\frac{1}{2}+Q}, \quad \psi_0 = \mathrm{e}^{\mathrm{i}\pi_2}\sqrt{1-2Q}, \quad \mathrm{e}^{\mathrm{i}(\pi_2-\pi_1)}\sqrt{\frac{1}{2}+Q}. \tag{5.2.10}$$

Here we use operators and coherent state eigenvalues interchangably. We then

find the following relation:

$$\begin{aligned}\hat{F}_\perp &:= \hat{F}_x + \mathrm{i}\hat{F}_y \\ &= \sqrt{2}\left(\psi_0^\dagger\psi_{-1} + \psi_1^\dagger\psi_0\right) \\ &= \mathrm{e}^{-\mathrm{i}\pi_1}\cdot 2\sqrt{1-4Q^2}.\end{aligned} \tag{5.2.11}$$

The phase angle measured in the experiment therefore corresponds to the Goldstone-field π_1 which is associated with the broken SO(2) symmetry. In the experiment $c_1 \sim c_0/100$ and since $Q \sim 1$ we also have $q \sim c_1$. The dispersion relations (4.2.43) therefore show that the mode associated with π_1 is of much lower energy compared to the total phase mode and it is therefore unsurprising that it is this mode which is predominantly occupied as the system approaches equilibrium.

The phase angle correlation function used in [13] is therefore just the $F_{11}(\mathbf{k}, t)$ introduced in (5.2.1). Since the scaling exponents for this correlator are the same as the one derived in 5.1.1 and 5.1.2 for any given case, we can draw the conclusion that for the case of quasi-particle number conservation the correlator is of the form

$$F_{11}(\mathbf{k}, t) \propto k_\Lambda(t)|\mathbf{k}|^{-(d+\frac{3}{2})} \quad , \tag{5.2.12}$$

where $k_\Lambda(t) \propto t^{-\beta}$. This means that in an effectively one-dimensional system we expect the phase angle correlators to scale in momentum as $|\mathbf{k}|^{-\kappa}$ with $\kappa_{\text{theo}} = 2.5$. The observed value is $\kappa_{\text{exp}} \approx 2.6$ and therefore the prediction based on the spin-1 model with quadratic Zeeman effect is much closer to the experimental results than the previously predicted $\kappa_{\text{U(N)}} = 2$ (see (5.3.4)).

Since the experimental setup involves a trapping potential, deviations from a perfectly free, one-dimensional system are to be expected and an effort could be made to quantify the effect that for example the degrees of freedom in the transverse directions have on this result.

5.3 U(3) - Scaling of Quasi-Particle Occupation Number

5.3.1 Spatio-Temporal Scaling

The same scaling analysis can be done for the U(3) symmetric Hamiltonian we studied in section 4.1. Previous versions of this analysis [12] were based on taking the more general U(N) model with N field components, then going to the large N limit and neglecting the linear Bogoliubov mode. In that case one finds a decoupling of the leftover modes and so both the 3-vertices as well as the 4-vertices that need to be taken into account involve only one mode interacting with itself. However, if restrict ourselves to N=3, this does not necessarily seem to be a valid approximation. Since the 3-vertex involve the interaction of two soft modes ($\omega_{\mathrm{i}} \propto |\mathbf{k}|^2$) with one Bogoliubov mode ($\omega_{\mathrm{B}} \propto |\mathbf{k}|$) and their spectra have different $|\mathbf{k}|$-dependence, the energy-conserving Dirac distribution appearing in the scattering integral I_3 does not exhibit a well defined scaling behaviour. Instead we can make the ansatz that at sufficiently low energies and late times the predominantly occupied modes are the soft ones. Integrating out the Bogoliubov mode turns this 3-vertex into a 4-vertex for the soft modes which scales exactly like the 4-vertex (4.1.53) that was already present in the Hamiltonian if one assumes that $\omega_{\mathrm{B}}(\mathbf{k}) \gg \omega_{\mathrm{i}}(\mathbf{k})$ for all total momenta $\mathbf{k}$ involved in the scattering of two soft modes via the Bogoliubov mode. We therefore find

$$\begin{aligned} I_4[f](\mathbf{k},t) &\propto \int_{\mathbf{q},\mathbf{p}} |\mathbf{k}|^4 f_{\mathrm{i}}^3(\mathbf{k}/\mathbf{q}/\mathbf{p},t)\; \delta(\omega_{\mathrm{i}}(\mathbf{k}) \pm \dots) + \text{similar terms} \\ I_4[f](\mathbf{k},t) &= s^{-\mu_4} I_4[f](s\mathbf{k}, s^{-\frac{1}{\beta}} t) \qquad\qquad \text{with} \quad \mu_4 = 2d + 2 - 3\frac{\alpha}{\beta} \quad . \end{aligned} \tag{5.3.1}$$

Together with the relations for conservation of number or energy we find the solution

$$\beta = \frac{1}{2} \qquad , \qquad \alpha = \frac{d}{2} \tag{5.3.2}$$

for number conservation and

$$\beta_4' = -\frac{1}{2} \qquad , \qquad \alpha_4' = -\frac{d+2}{2} \tag{5.3.3}$$

for energy conservation. These results are in agreement with the ones calculated in [12] for corresponding assumptions but differ significantly from the ones calculated for the spin-1 model. This is also the case for the momentum-dependence of the scaling function which for the U(N)-model is given by

$$\kappa_{\mathrm{U(3)}} = d + 1. \tag{5.3.4}$$

6 Conclusion and Outlook

We have seen that the general methods for the construction of low-energy effective theories in systems which exhibit spontaneous symmetry breaking lead to Lagrangians whose quantization requires a systematic treatment of constraints. It was shown that this constrained quantization produces results which are consistent with those obtained from standard Bogoliubov-theory.

The dynamics of a Spin-1 bose system were explored using these methods based on a model which involves only a spin-independent contact interaction, as well as a model which includes a spin-dependent interaction and a quadratic Zeeman-shift. The additional terms in the second model changed the internal dynamical symmetry group G from U(N) to U(1)xSO(2) and therefore the symmetry breaking pattern on which the construction of low-energy effective theories is based. The consequences were worked out in two different approaches. Firstly, the fundamental fields appearing in the full Lagrangian were replaced by a different parameterization, which is suited to the particular local ground state or phase around which we wanted to expand the theory. This parameterization was constructed from the symmetry breaking pattern and separated the field-degrees of freedom into Goldstone-fields and massive fields. The resulting Lagrangian was then quantized using the method of constrained quantization. Contrary to naive expectation, we have seen that it is not straightforward to predict the type of dispersion relations appearing in the free Hamiltonian from the number of fields of either type. For example, the U(N)-symmetric model involved one massive field, which corresponds to the total density, but there was no gapped modes in the final Hamiltonian. This is because the constraints appearing in the quantization procedure mixed the various fields in such a way, that the excitation modes correspond to combinations of the field fluctuations which are massless. This approach was compared to a general construction, which uses only symmetries and their braking pattern. In particular we have studied the case where a Lie group of internal symmetries G is spontaneously broken to a subgroup H. We have seen that Galilean spacetime symmetry combined with a gradient expansion and the properties of the coset space G/H greatly restrict the structure of effective Lagrangian at low energies. The number of quasi-particles could be deduced as well as their dispersion relations up to some free parameters, which depend on the details of the system instead of the symmetries only. We then showed that the two approaches match up if one employs a path integral formulation and integrates out the massive fields. It also became clear that the usefulness of the general approach greatly depends on the structure of the symmetries and the breaking pattern. For the U(N)-symmetric model both the quadratic part of the Hamiltonian, as well as the lowest order interactions originated from the same terms in the general Lagrangian and therefore had identical coefficients. This allowed us to obtain all of the relevant information for the application of the effective theory by only considering the Gaussian part of the full Lagrangian. For the U(1)xSO(2)-symmetric model this was not the case due to the symmetry group being abelian and it there-

fore turned out to be less useful to consider the general approach. However, we were able to predict that there must be two massless modes with linear dispersion relations in the broken-axisymmetry phase and therefore the four massive fields appearing in the full Lagrangian needed to contribute a gapped mode. This was indeed found to be the case by treating the full Lagrangian explicitly and, after integrating out the massive fields, we obtained the lowest order interactions. Combined with the spectrum, these constitute the relevant information required for a perturbative analysis of self-similar scaling at non-thermal fixed points.

Assuming that the non-equilibrium dynamics of the system in the vicinity of the broken-axisymmetry ground state is governed by a self-similar evolution of the statistical function for the Goldstone-fields, we were able to derive the scaling exponents from perturbation theory. The U(N)-model has been studied before in the large-N approximation and we found that for N=3 certain simplifications that have been made previously are problematic with respect to the scaling behaviour of the scattering integrals. However, we found that the statistical function of the soft modes behaves under reasonable assumptions just as predicted earlier. More interestingly, the model which includes spin-dependent interactions and the quadratic Zeeman shift did not require any of these assumptions in order to obtain a consistent set of equations for the scaling exponents. The results differed from the simpler model both in terms of the scaling in time and in terms of the momentum dependence of the scaling function.

Furthermore, we found that of the two massless modes predicted by the U(1)xSO(2)-symmetric model, one is much less energetic than the other if one considers the parameters of an experiment in which the non-equilibrium dynamics has been studied ([13]). The fact that the systems behaviour in the experiment seems to be well described by a single phase angle matches the expectation of the lower mode being occupied predominantly. We were able to identify the phase angle as the Goldstone-field associated with that mode and therefore our results for the scaling evolution should hold for the phase angle correlators that are measured in the experiment. Indeed, the momentum dependence of the scaling function is described by a scaling exponent κ, whose measured value of $\kappa_{\text{exp}} \approx 2.6$ is in good agreement with our prediction $\kappa_{\text{theo}} = 2.5$ for a one-dimensional system. This theoretical value is also an improvement over the prediction from the U(3)-symmetric model where $\kappa_{\text{U(3)}} = 2$. The remaining discrepancy may hint at corrections due to the fact that the experimental setup is not truly a one-dimensional free gas but involves a trapping potential.

Future research along this line could involve the application of the techniques which have been used in this work to the polar and ferromagnetic phases of the spin-1 bose gas and make corresponding measurements in order to see whether this kind of universal dynamics occurs here as well and is in accordance with the predicted scaling exponents. Also, it might be possible to include corrections to take into account the deviation from a perfectly one-dimensional system and thereby to obtain even better predictions.

7 Bibliography

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