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Universal Scaling in the (2+1) Dimensional Dual Sine-Gordon Theory Modelling Topological Defect Behaviour

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Universal Scaling in the (2+1) Dimensional Dual Sine-Gordon Theory Modelling Topological Defect Behaviour

In this thesis we build a magnetic monopole model for vortices in (2+1) dimensions and use the corresponding partition sum to form a statistically equivalent dual theory from the resulting action - the Sine-Gordon Theory. Thus we find a new dual action that is solely dependant on a scalar disorder field instead of regarding all vortex positions. Subsequently we use a fourth order approximation of the Sine-Gordon potential to deduce the universal scaling exponents of the vortex system. We find that it falls into the same universality class as the classical GPE description, having a universal scaling exponent of $\beta = \frac{1}{2}$. Furthermore we have a look at the approximations we made and check whether the calculation of the scaling for the full theory could lead to the experimentally predicted slower scaling with $\beta = \frac{1}{5}$. Additionally we discuss the overall behaviour of the Sine-Gordon potential, especially in the extreme temperature limits, comparing it to the Pair Collapse vortex state and the Onsager vortex state. We find a good agreement between our model and the theoretically expected phenomena.

Universelles Skalieren der (2+1) Dimensionalen Dualen Sine-Gordon Theorie zur Modellierung des Verhaltens Topologischer Defekte

In dieser Arbeit entwickeln wir ein magnetisches Monopol-Modell zur Beschreibung von Vortices in (2+1) Dimensionen und nutzen die dazugehörige Zustandssumme, um eine statistisch äquivalente duale Theorie - die Sine-Gordon Theorie - aus der resultierenden Wirkung zu bestimmen. Dadurch erhalten wir eine neue duale Wirkung die ausschließlich von einem skalaren "disorder field" abhängt anstatt alle Vortexpositionen zu berücksichtigen. Anschließend verwenden wir eine Näherung vierter Ordnung des Sine-Gordon Potentials, um die universellen Skalierungsexponenten des Vortexsystems zu finden. Wir stellen fest, dass es in dieselbe Universalitätsklasse wie die klassische GPE Beschreibung fällt und den gleichen universellen Skalierungsexponent $\beta = \frac{1}{2}$ besitzt. Weiterhin betrachten wir die getroffenen Näherungen und prüfen, ob die Berechnung des Skalierungsverhaltens der vollständigen Theorie die experimentell vorhergesagte Skalierung mit $\beta = \frac{1}{5}$ zeigen könnte. Schließlich diskutieren wir das allgemeine Verhalten des Sine-Gordon Potentials mit Fokus auf die extremen Temperaturbereiche und vergleichen es mit dem Paar-Kollaps sowie dem Onsager Vortexzustand. Wir finden eine gute Übereinstimmung zwischen unserem Modell und den theoretisch erwarteten Phänomenen.

Contents

Part I

Introduction and Theoretical Background

1 Introduction

Quantum many-body systems quenched far from equilibrium are a topic in which a lot of new findings have been made in the last few years. They can be found in many different fields of physics, describing for example the post-inflationary early universe, the quark-gluon matter created in heavy-ion collisions or the evolution of ultracold atomic systems following a sudden quench [1].
Still there are many phenomena in this field that are not yet understood very well, one of them being the universal scaling of topological defect (vortex) behaviour during a disorder to order transition in ultracold atomic gases (see figure 1.1).

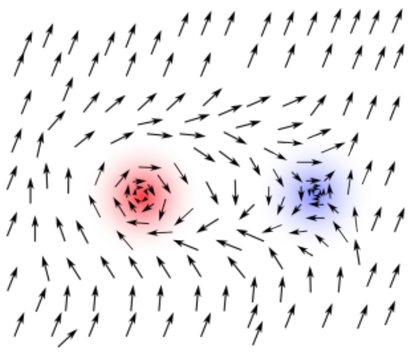
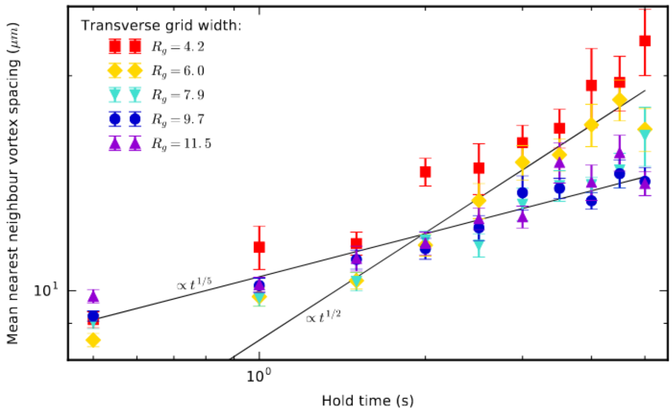


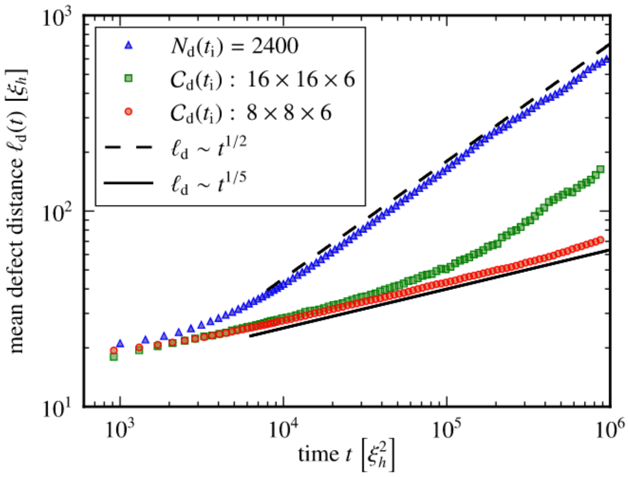
Figure 1.1: Perturbance of order due to a vortex-anitvortex pair.
Figure taken from [2]

Recently it has been found both in experiments [3] and simulations [4] that these quantum vortex transitions can have two different kinds of scaling. Depending on the initial conditions they can either approach a Gaussian fixed point leading to the scaling exponent $\beta = \frac{1}{2}$ or they can (at first) approach an anomalous fixed point with a much slower scaling of $\beta = \frac{1}{5}$ (see figure 1.2).
However this behaviour is only suspected to be sourced by three body recombinations of the vortices and can not be analytically explained yet. Thus the goal of this thesis is to find a promising ansatz to be able to establish the analytical reason for and to gain further insight into the processes.

One option that could be a starting point for this goal is to construct an effective field theory for the vortices and calculate the scaling behaviour from it [5], [6]. This method has the disadvantage that it regards the information on all vortex positions at all times making the calculation and renormalization very difficult for higher vortex numbers.



(a) Measurement of the universal scaling for different vortex grid sizes.



(b) Simulation results showing the scaling for different vortex grid sizes.

Figure 1.2: (a) Plot of the mean nearest neighbour inter-vortex spacing as a function of time. At early times the scaling exponent is $\beta = \frac{1}{5}$ (anomalous fixed point) for all grid widths but for the finer grids it transitions to $\beta = \frac{1}{2}$ (Gaussian fixed point), while this transition sems to be suppressed for the more coarse grids. Figure taken from [3]
(b) Mean vortex distance as a function of time for different initial vortex configurations: a random distribution of 2400 vortices and antivortices (blue) and irregular square lattices of $8 \times 8 \times 8$ (red) and $16 \times 16 \times 16$ (green) non elementary vortices (winding number ± 6). The random distribution scales with $\beta = \frac{1}{2}$ (Gaussian fixed point) while the square lattice distributions first scale with $\beta = \frac{1}{5}$ (anomalous fixed point) with the bigger lattice distribution later also approaching the Gaussian fixed point. Figure taken from [4]

Another option is to make a hydrodynamic approach to describe the vortex behaviour by considering the superfluid flow [7], but here the computations also get quite difficult, and long.

Thus in this thesis we choose a different approach by carrying out a duality transformation, which is a tool to find a statistically equivalent system reducing it to a mean field theory and therefore losing the - for the universal scaling unnecessary - information on the vortex positions [8],[9]. After giving a broad overview on the theoretical background (chapter 2) we then need to construct a full model for the vortices as a starting point for the calculations. In our case we consider a time evolution of vortices on a plane working in (2+1) dimensions. Next we use the concept of magnetic monopoles to source the vortex fields and derive the corresponding action (chapter 3). Of the resulting action we deduce the partition sum of the system and perform a so called duality transformation. There from we receive the statistically equivalent dual Sine-Gordon action that depends solely on a disorder field ϕ (chapter 4). We can then utilize it by computing the universal scaling of the dual theory, which corresponds to the scaling of the starting model. In this step we approximate the potential to fourth order (chapter 5). Finally we discuss the physical interpretation of our findings and the approximations we made (chapter 6,7).

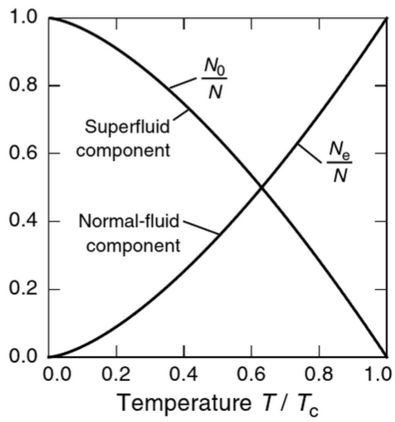
2 Theoretical Background

In the present chapter the theoretical background of the calculations in this thesis is presented, starting with a short summary about Bose Einstein condensates (BECs) and vortices and explaining the mechanism of a quantum phase transition. An example for such a phase transition are the so called Brenzinsky-Kosterlitz-Thouless and Mott Transitions, which are explained in the second and third section of this chapter. Finally the general functionality of universal scaling behaviour and non thermal fixed points is discussed in the last section.
A more complete overview on the physical background of these topics can be for example found in A. Altland et al. *Condensed Matter Field Theory* [10] and X.G. Wen *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* [11].

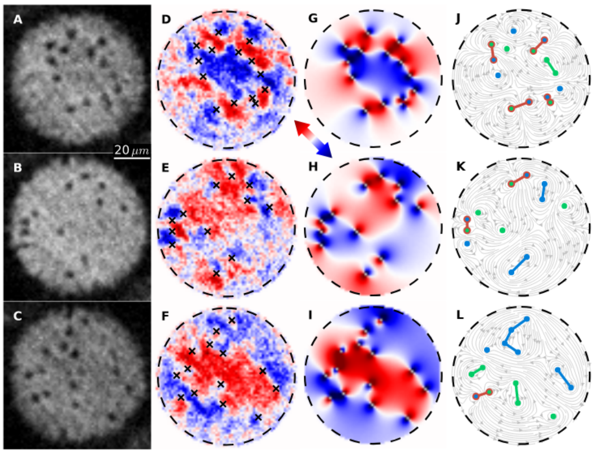
2.1 Bose-Einstein Condensates : Vortices and Quantum Phase Transitions

If a dilute ideal Bose gas is cooled down to a critical temperature (T_C) close to absolute zero nearly all of the bosons are locked together into the lowest quantum state making them completely undistinguishable. Therefore in this state they can all be described by one single wave function and thus Bose Einstein condensation can be seen as a disorder to order transition.

This new state has some very special characteristics resulting from it. They are superfluidity, superconductivity, supersolidity and coherence over macroscopic distances. In the following we a mostly interested in the superfluidity, since it is needed to explain quantum vortices in BECs. The difference between a normal fluid and a superfluid is that a superfluid loses all of its internal friction. Therefore there is no energy dissipation generated by the particles of the flow scattering off imperfections and concerting their energy into the creation of elementary excitations. In a superfluid the flow is dissipationless simply put because the kinetic energy carried by the particles is too low to be able to create the temperature related energetically high lying excitations [12].
As a special phenomenon of superfluidity the occurring vortex excitations (see figure 2.1(b)) are always quantized. This means that the phase configurations only change by multiples of 2π as one moves around the vortex center. In the following we will show why this is the case by following the calcualtion in [12].



(a) Ground state population for different temperatures



(b) Vortices in a Bose Einstein Condensate

Figure 2.1: (a) Proportions of the ground state ($\frac{N_0}{N}$) and the excited states ($\frac{N_e}{N}$) at different temperatures relative to the condensation temperature $\frac{T}{T_C}$ showing Bose Einstein condensation. Figure taken from [12].
(b) Vortices (dark spots) in the optical density images of a BEC in the pair collapse, random and clustered regimes (A,B,C). The corresponding Bragg spectroscopy signals (D,E,F) and computed projection of the velocity field (G,H,I) with the colors showing the direction of the projection of the superflow according to the arrow. The last column (J,K,L) shows the classification of the vortices: vortices are marked blue and antivortices green with clusters shown by lines of the same color and vortex-antivortex pairs shown by red lines. Streamlines of the computed flow are shown in gray. Figure taken from [3].

Using the fact that we can describe the condensate by its unitary wave function depending on the local particle density $n(r,t)$ and the local phase $S(r,t)$ we get:

$$\psi_0(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp(iS(\mathbf{r},t)) \tag{2.1}$$

Furthermore we see that the particle current density then takes the form

$$\mathbf{j}(\mathbf{r},t) = -\frac{i\hbar}{2m}(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*) = n(\mathbf{r},t)\frac{\hbar}{m}\nabla S(\mathbf{r},t)$$

which means that the superfluid velocity is sourced by the phase field $S(r,t)$:

$$\mathbf{v}_s(r,t) = \frac{\hbar}{m}\nabla S(\mathbf{r},t) \tag{2.2}$$

If we check for the classical rotational part of this velocity field we immediately see that it must be zero.

$$\nabla \times \mathbf{v}_s(\mathbf{r},t) = \frac{\hbar}{m}\nabla \times \nabla S(\mathbf{r},t) = 0 \tag{2.3}$$

On the other hand calculating the circulation κ by integrating the curl of the velocity over a closed circular area A we get with Stokes' Theorem :

$$\kappa = \int_A \nabla \times \mathbf{v}_s \mathrm{d}\mathbf{f} = \oint_{\partial A} \mathbf{v}_s \mathrm{d}l = \oint_{\partial A} \frac{\hbar}{m} \nabla S(\mathbf{r}, t) \mathrm{d}l = \frac{\hbar}{m} \delta S(\mathbf{r}, t) \tag{2.4}$$

Since the wave function has to be unique as stated in the beginning the phase difference $\delta S(r,t)$ can only take integer mutiples of 2π for a complete circle. Thus we are left with quantized vortices with the circulations $\kappa = \frac{2\pi\hbar}{m} n$ for $n = \pm 1, 2, 3, \ldots$.

The last concept we want present in this chapter is the concept of a quantum phase transition. In comparison to a classical system where the entropy goes to zero for $T = 0$ so that no classical phase transition (e.g. from solid to fluid) can happen, a quantum phase transition takes place at temperatures close to zero going from one quantum phase of the system to another one. Generally phase transitions are classified as first or second (or third...) order depending on the order of the derivation going to zero.
In figure (2.2) a quantum phase transition of second order can be seen describing a transition from a disordered to a ordered state : At $T = 0$ there is only the pure quantum phase transition taking place at the quantum critical point (QCP) but for higher temperatures still close to zero the classical phase transitions and quantum phase transition are both present.

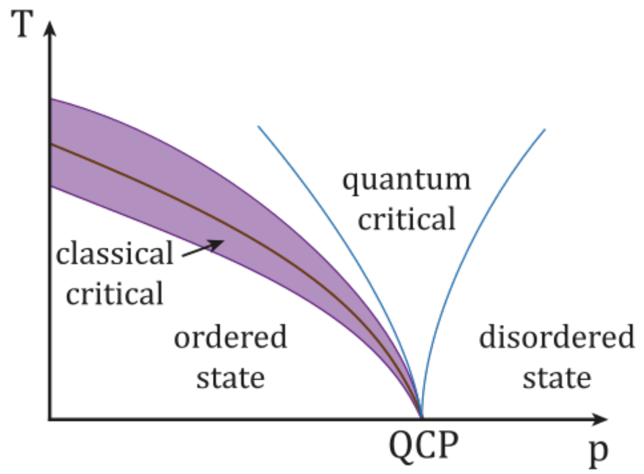


Figure 2.2: Phasediagram with temperature T (y axis) over pressure p (x axis) showing the concept of a quantum phase transition. The different states are seperated by lines indicating phase transitions which meet at the quantum critical point (QCP) for $T = 0$. Figure taken from [13]

Subsequently we will look at two types of vortex quantum phase transitions : the transition to the pair collapsed vortex state (Berezinsky–Kosterlitz–Thouless (BKT) Transition) and the transition to the clustered vortex state (Onsager state).

2.2 Berezinsky–Kosterlitz–Thouless Transition

The first quantum phase transition we want to have a close look at is the Berenzinsky-Kosterlitz-Thouless (BKT) Transition. After predicting this phenomenon Kosterlitz and Thouless received the Nobel Prize in physics 'for theoretical discoveries of topological phase transitions and topological phases of matter' in 2016, because of the BKT Transition being the first topological phase transition meaning that it is generated only by topological defects (vortices).
The model this transition was predicted for is the so called two dimensional XY-Model, a model consisting of spins allowing for 2d quantized vortices on a square lattice. In this model at low temperatures the system behaves like a dilute gas of tightly bound vortex-antivortex pairs while at higher temperatures above a critical temperature T_C the pairs unbind making the system act like a vortex plasma.
Special about this is that there is no classical order parameter (like e.g. the magnetization) for the BKT-Transition going to zero at the critical temperature (as a result from being a topological phase transition) and no spontaneous symmetry breaking, making it a phase transition of infinite order ([14]).

For a two dimensional Bose Gas the unbinding of the vortex-anti vortex pairs at the BKT Transition destroys the superfluidity of the system [15], this concept is shown in figure (2.3):

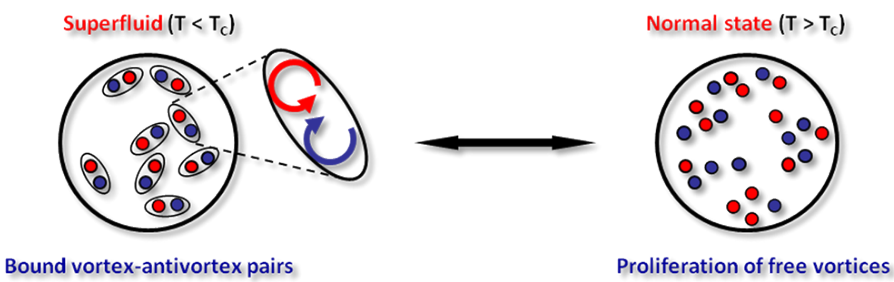


Figure 2.3: Mechanism of the BKT-Transition: On the left side (temperature below critical temperature $T < T_C$) the vortices are bound into vortex-anitvortex pairs. Nearing the transition point there is a higher density of the pairs and the average pair size diverges. On the right side (temperature above the critical temperature $T > T_C$) the vortices are free and superfluidity is destroyed. Figure taken from [16]

In this thesis we are going to construct a statistical model describing the vortex behaviour, and amongst other things checking whether it is able to depict the vortex mechanism near this phase transition.

2.3 Einstein-Bose Condensate: Onsager vortex state

Normally quenches (which means rapidly/instantaneously changing a parameter) in isolated systems lead to increased entropy at equilibrium by increasing the energy per particle and making the system more disordered. If a system has a limited phase-space however it will at some point get more ordered when adding energy, making the entropy decrease. A decreasing entropy for increasing energy is only possible for a state with negative absolute temperature. To be able to realize such a thermodynamic state a system needs to be isolated because if it came in contact with a positive temperature state the energy would spontaneously flow to it [17].

For the two dimensional point vortex model this is possible because the point vortices themselves have no inertial kinetic energy so that the phase space is determined entirely by the finite area available to the point vortices, making it limited [17]. As a result at negative temperatures vortices and anti-vortices separate creating clusters that correspond to the highest accessible energy states of the vortex degrees of freedom. This so called 'Onsager vortex state' named after Lars Onsager, who first predicted it in the 1940s, therefore decreases the entropy forming a 'Einstein Bose condensate' [18], [19]. In figure (2.4) the whole behaviour of the point vortex model going from a BEC to a EBC for increasing energy with both transitions is shown, the corresponding measurements can be seen in figure 2.1 (b).

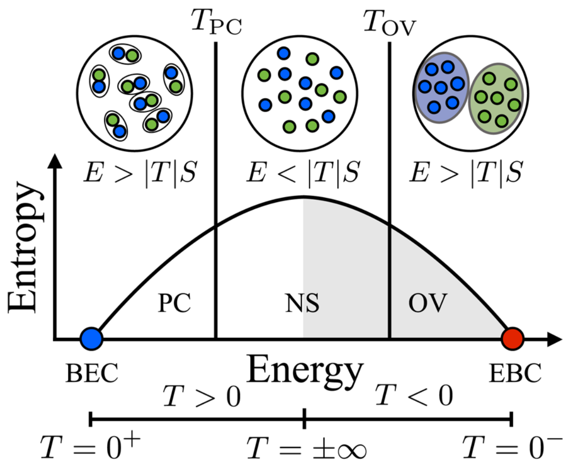


Figure 2.4: Plot of entropy versus energy for the point-vortex model: Starting from a BEC with tightly bound vortices (pair collaps state (PC)) at $T = 0$ the entropy increases at higher temperatures converting the system to a normal state (NC) at a critical temperature T_{PC} with free vortices (BKT-Transition). By increasing the energy further a negative temperature state with decreasing entropy is created. At a second critical temperature T_{OV} the vortices therefore transition to the Onsager vortex state and a Einstein-Bose condensate (EBC) is formed. Figure taken from [17]

2.4 Universal scaling behaviour: Non-thermal fixed points

The last phenomenon we are going to about is universal scaling close to non thermal fixed points (NTFPs). The present chapter will be a short summary of this mechanism following the explanations in C.M. Schmied et al. *Non-thermal fixed points: Universal dynamics far from equilibrium* [1].

The concept of universal scaling behaviour near non-thermal fixed points is based on the idea of a renormalization group flow. One can observe that near phase transitions correlations show self similar behaviour. This means that if we rescale our system by a factor s at a fixed point the rescaled correlations $C(x, s) = s^\gamma f(x/s)$ do not depend on s but only on a universal scaling exponent γ and a universal scaling function f.
If we take the time as the scaling parameter s we get a scaling in space and time characterized by:

$$C(x, t) = t^\alpha f(t^\beta x) \tag{2.5}$$

The two scaling exponents α and β together with the scaling function f determine the universality class of the system. As a result the universality class can be used to compare very different physical systems and categorize their behaviour near a non thermal fixed point. To be able to undergo such universal scaling and reach a non thermal fixed point on its way to equilibrium a system has to have extreme out of equilibrium initial conditions. The mechanism of this process is shown in figure 2.5.

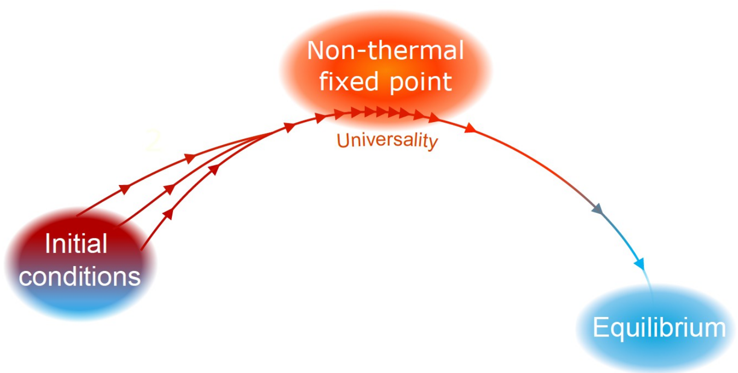


Figure 2.5: Mechnism of a non-thermal fixed point: On its way to equilibrium a system can approach a non thermal fixed point, that depends on the initial conditions. Near this fixed point the system critically slows down so that correlation functions show universal scaling behaviour in space and time. The scaling can be characterized by a universal scaling function and its scaling exponents. Figure taken from [1]

In our case we are interested in the universal scaling of a dilute Bose gas. This system can be brought into extreme out of equilibrium conditions by a strong cooling quench, which leads to a momentum distribution that suddenly drops at a maximum momentum scale and is therefore strongly overpopulated below this scale. Thus an inverse particle transport to low momenta takes place while the energy flows to high momenta. For both directions there are different universal scaling exponents α and β characterizing the scaling of the momentum distribution at a reference time t_{ref}:

$$n(t, \mathbf{p}) = \left(\frac{t}{t_{ref}}\right)^{\alpha} n\left(t_{ref}, \left[\frac{t}{t_{ref}}\right]^{\beta} |\mathbf{p}|\right) \tag{2.6}$$

In figure 2.6 a sketch of the described scaling process of the momentum distribution for a dilute Bose gas can be seen.

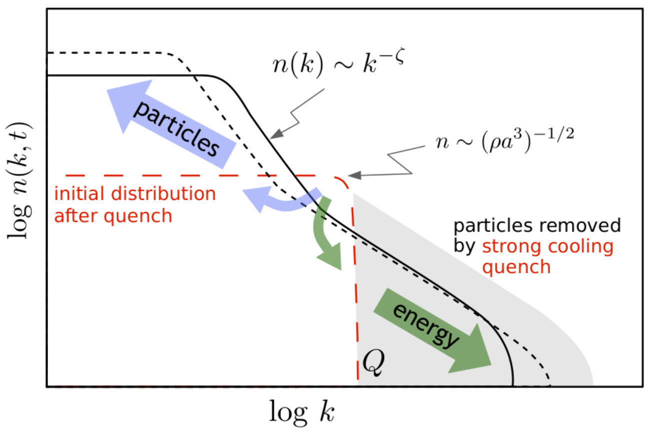


Figure 2.6: Concept of self-similar scaling behaviour near a non thermal fixed point: Double logarithmic plot of a single particle momentum distribution $n(k, t)$ for a Bose gas. The red dashed line shows the out of equilibrium initial distribution after a cooling quench, while the solid and dashed black lines show the time evolution of the system. The redistribution of the particles in momentum space, that can be described by self-similar scaling with the universal scaling exponents α and β, is symbolized by the two arrows: Particles are transported towards low momenta while energy goes to large momenta with the two directions being described by different scaling exponents. As a result the infrared transport conserves particle number and the high momentum transport conserves energy. Figure taken from [1]

In this thesis we are going to determine the universal scaling exponents of a Bose gas, that is quenched to far from equilibrium initial conditions so that vortices are generated and than experiences a universal scaling toward the ordered phase of pair collapsed vortices.

Part II

Model Building and Duality Transformation

3 Magnetic Monopole Flux as a Model for Vortex Behaviour in 2+1 D

In the following chapter we are building our theoretical model to describe the vortex behaviour starting from explaining the ansatz and showing the background of two-dimensional electrodynamics. From the starting point of the magnetic flux we then calculate the dual electromagnetic tensor (or here vector) and construct the action from the corresponding Lagrangian.

3.1 Motivation as Model

Our aim is to find a model for the behaviour of vortices in a Bose Einstein Condensate in only two spacial dimensions. Another physical subject area from which we are familiar with vortex-fields, is electrodynamics. It is well known from the Maxwell Equations (table 3.1) that a change in the magnetic field in time or a magnetic flux result in an electric vortex-field and vice versa.

Since electrodynamics are a well understood quantizable gauge theory the thought to use this as our model seems reasonable. Now we only need to construct an electrodynamic picture in which the electric field represents our vortex field with a magnetic flux sourcing it. To do this we use Dirac monopoles or rather Dirac strings that lie in an imaginary third space dimension and generate vortices in their intersection with our world-plane.

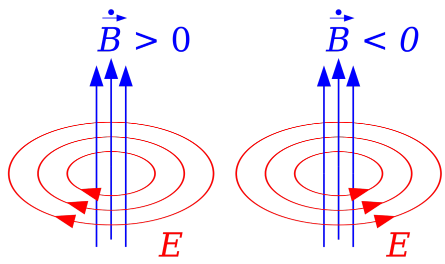


Figure 3.1: Induction of a two-dimensional electric vortex field (red) by a change in the magnetic field (blue) as a model for vortex and antivortex. Figure taken from [20]

In the following calculations we are going to start from the basis of 2+1 dimensional electrodynamics and magnetic monopoles to construct the magnetic flux sourcing our vortices. From there we will calculate the action of the system to get a model description of our situation.

3.2 Electrodynamics in 2+1 Dimensions

The Maxwell equations in three space dimensions are possibly the best known equations in physics. But since we want to describe a two-dimensional situation we need to have a look at their two-dimensional form.
The electric field is supposed to represent the vortex field, therefore it has to lie completely in the two space dimensions of the system. Additionally the magnetic field has to be orthogonal to the electric field to be able to induce it. To make this possible we define a third 'imaginary' dimension in which the magnetic field lives (see figure (3.2)). This 'imaginary' direction is not a physical space dimension but only a conceptual thought that enables us to construct an electromagnetic system representing the two-dimensional vortices.

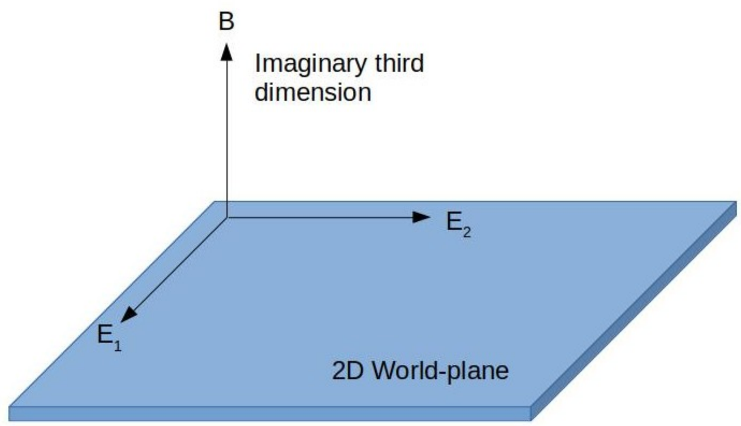


Figure 3.2: Construction of our system as a two-dimensional world-plane containing the electric field with the orthogonal magnetic field in an 'imaginary' third dimension

This concept reduces the electric field and the electric flux to two-dimensional vectors while the magnetic field and the magnetic flux are reduced to scalars. If we apply this to the Maxwell equations the differential operations have to be adapted accordingly.
In table (3.1) the Maxwell equations in Gaussian units for two and three dimensions are presented. To shorten the calculations we will later on use the definition of the perpendicular gradient :

$$\nabla_\perp = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix} \tag{3.1}$$

Law	3D-Maxwell equations	2D-Maxwell equations
Coulomb	$\nabla \cdot \mathbf{E} = 4\pi\rho_E$	$\nabla_{x_1,x_2} \cdot \mathbf{E} = 4\pi\rho_E$
Ampere	$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j_E} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$	$\begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} B = \frac{4\pi}{c}\mathbf{j_E} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$
Gauss	$\nabla \cdot \mathbf{B} = 4\pi\rho_M$	$(\partial_{x_3} B = 4\pi\rho_M)$
Faraday	$\nabla \times \mathbf{E} = -\frac{4\pi}{c}\mathbf{j_M} - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$(\partial_x E_2 - \partial_y E_1) = -\frac{4\pi}{c} j_M - \frac{1}{c}\frac{\partial B}{\partial t}$

Table 3.1: Comparison of the Maxwell equations in 3D and 2D (Gaussian units)

Furthermore we need to have a look at the electromagnetic field tensor. In three dimensions (in Gaussian units as well) it takes the form :

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{3.2}$$

To get the full Maxwell equations we also need the dual field tensor:

$$\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \tag{3.3}$$

If we restrict it to two space dimensions we simply have to cut off the last column and row (for further theoretical background see D. Boito et al *On Maxwell's electrodynamics in two spatial dimensions* [21]).

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -B \\ E_2 & B & 0 \end{pmatrix} \tag{3.4}$$

The changes are a bit bigger for the dual field tensor: In two dimensions it is reduced to a vector - we will use this fact in the following chapters to our advantage since we can utilize it to simplify the action of the system.

$$\tilde{F}_\lambda = \frac{\epsilon_{\mu\nu\lambda} F^{\mu\nu}}{2} \tag{3.5}$$

From the electromagnetic field tensor we can calculate the Lagrangian and the action the way we are used to from three dimensions as constructed:

$$\mathcal{L}_A = \frac{K}{4} F^{\mu\nu} F_{\mu\nu} \tag{3.6}$$

$$S = \frac{K}{4} \int \mathrm{d}x^3 F^{\mu\nu} F_{\mu\nu} \tag{3.7}$$

3.3 Magnetic Monopoles sourcing a Magnetic Flux

To use this electrodynamic two-dimensional world as a model to describe vortices we need it to induce symmetric two-dimensional vortex fields. Therefore according to the Maxwell equations (see table 3.1) we nee a change in the magnetic field in time or a magnetic flux. The easiest way to create this is - parallel to the way we would do this for an electric flux - to take magnetic 'point-charges' that move through orthogonal to the plane (see figure 3.3).
From here on we will call the two dimensions in which our vortices live the 'world plane' to emphasise that these are the only two space dimensions that physically exist and in which the time evolution of our 2+1 dimensional model takes place.

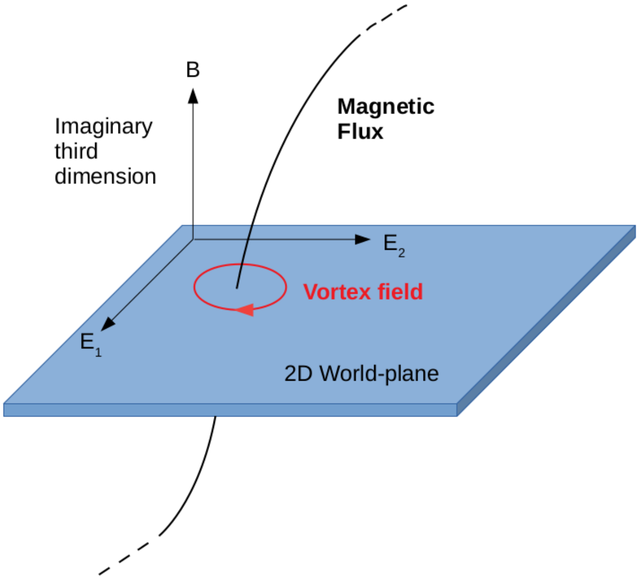


Figure 3.3: Concept sketch of the model : magnetic flux going through perpendicular to the 2D world-plane and sourcing a vortex field - for the two-dimensional world the flux therefore only has a contribution in the third imaginary direction

For this purpose we need to introduce magnetic monopoles ('point charges') into our theory. Therefore we symmetrize the Maxwell equations so that they contain a magnetic flux and a magnetic density 3.1 (for further background on this see for example [22]). Now we can construct a magnetic flux from moving magnetic point charges in the three-dimensional space. When the flux goes through our world plane it switches on an induced electric vortex or antivortex field depending on the flux direction.
Additionally the quantisation of the magnetic charge ensures the quantisation of the vortex field it sources. Thus we naturally get quantum vortices from this model, which is exactly what we need for a description of the vortices in a Bose-Einstein Condensate.

3.4 Reduction of the Magnetic Flux

We start with the magnetic flux $\mathbf{j}_M(x,t)$ for moving magnetic point charges $q_m^{(i)}$ (the Dirac monopoles) at the positions $\mathbf{x}^{(i)}(t)$. The subscript m labels the variables belonging to the monopoles with the different monopoles counted by the index i. The positions $\mathbf{x}^{(i)}(t)$ describe the world-lines of the monopole in the three-dimensional space we constructed from our two-dimensional world-plane $\mathbf{x}^{(i)}{}_\perp = (x^{(i)}{}_1, x^{(i)}{}_2)$ and the imaginary third dimension $x^{(i)}{}_3$ (see figure 3.3).

$$\mathbf{j}_M(\mathbf{x},t) = \sum_i q_m^{(i)} \mathbf{v}^{(i)}(t)\delta^{(3)}\left(\mathbf{x} - \mathbf{x}^{(i)}(t)\right) \qquad \mathbf{v}^{(i)}(t) = \partial_t \mathbf{x}^{(i)}(t) \tag{3.8}$$

To simplify calculations we reduce our situation to only one monopole and set the monopole velocity v_{m3} to be strictly positive since the sign of the magnetic charge q_m already creates the two different vortex field directions. Furthermore we set the intersection of the monopole with the world-plane to $t = t_0$ so that we can use $\mathbf{x}_m(t) = \mathbf{v}_m(t)(t-t_0)$ and $\mathbf{x}_m(t=t_0) = (x_{m1}, x_{m2}, 0)$ to reduce the delta distribution to 2D.

$$\begin{aligned}
\mathbf{j}_M(\mathbf{x},t) &= q_m \mathbf{v}_m(t)\delta^{(3)}\left(x - x_m(t)\right) \\
&= q_m \mathbf{v}_m(t)\delta\left(x_3 - v_{m3}(t)(t-t_0)\right)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{x}_{m\perp}(t)\right) \\
&= q_m \frac{\mathbf{v}_m(t)}{|v_{m3}(t)|}\delta\left(t - t_0 - \frac{x_3}{v_{m3}(t)}\right)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{x}_{m\perp}(t)\right)
\end{aligned}$$

Since we want to describe the behaviour on the 2D world-plane we have to set $x_3 = 0$ (while we can - without loss of generality - use $v_{m3} \neq 0$) which reduces our flux to:

$$\mathbf{j}_M(\mathbf{x},t) = q_m \frac{\mathbf{v}_m(t)}{|v_{m3}(t)|}\delta\left(t - t_0\right)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{x}_{m\perp}(t)\right) \tag{3.9}$$

The first delta function now sets the time to t_0 so that we can use the fact that we constructed the Dirac string to be perpendicular to the world-plane at the intersection which we set to $t = t_0$. Being perpendicular only the v_{m3} component of the monopole movement is non-zero : $\mathbf{v}_{m\perp}(t = t_0) = 0$

As a result the only non-zero component of $\mathbf{j}_M$ is j_{M3} and the $\frac{\mathbf{v}_m(t)}{|v_{m3}(t)|}$ factor is reduced to one because of v_{m3} being positive and can be dropped. This leaves us with the simplified scalar magnetic flux, going into the imaginary third dimension as we expected from the 2D Maxwell equations:

$$j_M(\mathbf{x},t) := j_{M3}(\mathbf{x},t) = q_m \delta\left(t - t_0\right)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{x}_{m\perp}(t)\right) \tag{3.10}$$

or for multiple monopoles

$$\boxed{j_M(\mathbf{x},t) = \sum_i q_m^{(i)} \delta\left(t - t_0^{(i)}\right)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{x}_{m_\perp}^{(i)}(t)\right)} \tag{3.11}$$

Here we can see that in the 2D world-plane the x_3 component of the magnetic flux j_M seems to be equivalent to a 2+1 dimensional magnetic monopole density: It describes magnetic point charges that pass through at $t = t_0^{(i)}$ at $\mathbf{x}_{m_{i_\perp}}(t = t_0^{(i)})$ and are moving in the imaginary third dimension, which is exactly what we wanted to construct (see figure 3.3).

3.5 Construction of the Electromagnetic Field Vector

From the magnetic flux we constructed in the last section we can now calculate the electromagnetic field tensor by using the fourth Maxwell equation (see table 3.1), which we need later on to be able to calculate the action of the system.

Since the electromagnetic field tensor $F^{\mu\nu}$ is traceless and antisymmetric, there can only be three independent components in 2+1 dimension. Therefore we first reduce $F^{\mu\nu}$ to the dual field vector $\tilde{F}_\lambda$ to simplify the calculations:

$$\tilde{F}_\lambda = \frac{\epsilon_{\mu\nu\lambda} F^{\mu\nu}}{2} \tag{3.12}$$

Using the definition of $F^{\mu\nu}$ in two spatial dimensions (3.4) we get :

$$\tilde{F}_\lambda = (-B, E_2, -E_1) \tag{3.13}$$

Taking the divergence of this we immediately recognize the fourth Maxwell equation, which gives us the connection between our new electromagnetic field vector and the magnetic flux we constructed above.

$$\partial^\lambda \tilde{F}_\lambda = -\frac{1}{c}\frac{\partial B}{\partial t} - \frac{\partial E_2}{\partial x} + \frac{\partial E_1}{\partial y} = \frac{4\pi}{c} j_M \tag{3.14}$$

Next we want to express this in terms of an electromagnetic potential we will call V_m. Since we have no electric source (for $j_E = 0$ and $\rho_E = 0$) we know that $\nabla E = 0$, therefore we can define

$$\mathbf{E} = -\nabla_\perp V_m = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} V_m \tag{3.15}$$

for the electric field. Making sure that the second Maxwell equation is satisfied we get:

$$B = \partial_{ct} V_m \tag{3.16}$$

From this we can write the electromagnetic field vector as :

$$\tilde{F}_\lambda = (-B, E_2, -E_1) = (-\partial_{ct} V_m, -\partial_x V_m, -\partial_y V_m) = -\partial_\lambda V_m \tag{3.17}$$

Plugging this into equation 3.14 we finally arrive at :

$$\boxed{-\frac{4\pi}{c}j_M = \partial^\lambda \partial_\lambda V_m} \tag{3.18}$$

This is the 2+1 dimensional wave equation for V_m which is solved by the respective retarded Green's Function.

$$G_R^{(2+1)}(ct, \mathbf{x}) = \frac{\theta(t - \frac{|\mathbf{x}|}{c})}{2\pi c\sqrt{(ct)^2 - (\mathbf{x})^2}} \tag{3.19}$$

We get the expression:

$$V_m(ct, \mathbf{x}) = \frac{-4\pi}{c}\int \mathrm{d}ct'\mathrm{d}^2x' G_R^{(2+1)}(ct - ct', \mathbf{x} - \mathbf{x}')j_M(ct', \mathbf{x}') \tag{3.20}$$

$$= \frac{-4\pi}{c}\int \mathrm{d}ct'\mathrm{d}^2x' \frac{\theta(t - t' - \frac{|(\mathbf{x}-\mathbf{x}')|}{c})}{2\pi c\sqrt{(ct - ct')^2 - (\mathbf{x} - \mathbf{x}')^2}} j_M(t', \mathbf{x}') \tag{3.21}$$

This can be combined with the expression for $\tilde{F}_\lambda$ results in :

$$\tilde{F}_\lambda(ct, \mathbf{x}) = \frac{-4\pi}{c}\partial_\lambda \int \mathrm{d}ct'\mathrm{d}^2x' G_R^{(2+1)}(ct - ct', \mathbf{x} - \mathbf{x}')j_M(ct', \mathbf{x}') \tag{3.22}$$

And if we substitute ct with t we finally get:

$$\boxed{\tilde{F}_\lambda(\mathbf{x}^\mu) = (-4\pi)\partial_\lambda \int \mathrm{d}t'\mathrm{d}^2x' G_R^{(2+1)}(t - t', \mathbf{x} - \mathbf{x}')j_M(t', \mathbf{x}')} \tag{3.23}$$

This shows that the dual electromagnetic field tensor is a 2+1 dimensional gradient field, sourced by a classical electrodynamic potential that is defined by the convolution of a Greens function and a point charge density (which in our system is the magnetic flux j_M that takes the place of a magnetic monopole density - see chapter 3.5).

3.6 Calculation of the action

In this chapter we want to use the dual electromagnetic field vector F_λ which we have constructed above to calculate the action. To do this we first write the corresponding Lagrangian with the constant K :

$$\mathcal{L}_A = \frac{K}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.24}$$

For the following derivation it is useful to express the Lagrangian in terms of the dual electromagnetic field vector:

$$\mathcal{L}_A = \frac{K}{2}\widetilde{F}_\mu \widetilde{F}^\mu \tag{3.25}$$

We can check this easily by plugging in the definition of F_μ and using the fact that F_μ is antisymmetric:

$$\begin{aligned}
\mathcal{L}_A &= \frac{K}{2}\widetilde{F}_\mu \widetilde{F}^\mu = \frac{K}{8}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}\epsilon^{\mu\kappa\delta}F_{\kappa\delta} \\
&= \frac{K}{4}\frac{1}{2}\epsilon_{\mu\nu\lambda}\epsilon^{\mu\kappa\delta}F^{\nu\lambda}F_{\kappa\delta} = \frac{K}{4}\frac{1}{2}(\delta^\kappa_\nu\delta^\delta_\lambda - \delta_n u^\delta \delta^\kappa_\lambda)F^{\nu\lambda}F_{\kappa\delta} \\
&= \frac{K}{4}F^{\mu\nu}F_{\mu\nu}
\end{aligned}$$

From this we can write the action of our system. Since we want to transform this action to a simpler form later we don't need to plug the result for $\widetilde{F}_\mu$ into the integral.

$$\boxed{S = \frac{K}{2}\int \mathrm{d}^{(2+1)}x \widetilde{F}_\mu \widetilde{F}^\mu} \tag{3.26}$$

This action can be used within a path-integral approach to derive an effective field theory for the vortex dynamics. The problem of the described approach is that the system is dependent on the positions of all vortices at all times, which means that for more that two or three vortices there are many degrees of freedom making the calculations very long and hard to renormalise (for example see [6]).
To avoid this problem we changed to a statistical piont of view losing the information about the single vortex positions while working with the grand canonical partition function following the results of H.Kleinert et al. in [23].

4 Duality Transformation - Sine-Gordon Theory

In the following part of the thesis we are going perform a Duality Transformation to transfer our system into a statistical equivalent. The starting point is the classical electromagnetic action from chapter 3 and after the transformation we obtain the dual action which is a Sine-Gordon Theory.

4.1 Concept of a Duality Transformation

In the last chapter we constructed an electromagnetic model to describe the physics of vortices in a Bose Einstein Condensate. As a result we now have an action, which depends on the quadratic dual electromagnetic field tensor or in our case vector, that combines the information of all vortex position at all times. As concluded in the last section we can look at this system from a statistical point of view by writing down the corresponding grand canonical partition function, because we only want to make a statement about the scaling behaviour.

The idea of the following 'Duality Transformation' is to work with this partition function to transform it into a more practical form, that now only depends on a so called 'disorder field' (see figure 4.1). From the new but equivalent partition function we then extract the action which is - because of this method - statistically the same as the original one. Therefore it has the same scaling properties leading us to our universal scaling behaviour without having to deal with the many degrees of freedom of the single vortex movements.

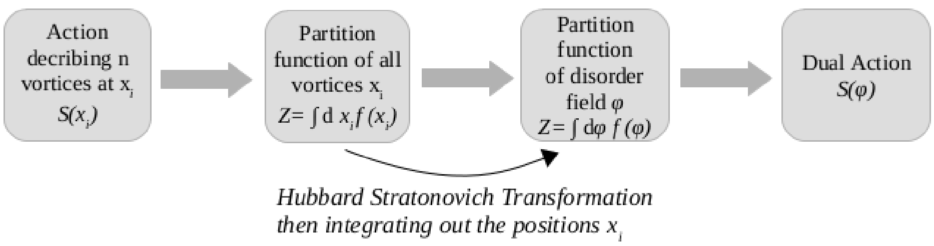


Figure 4.1: Concept of the duality transformation we will perform in the following chapters using the partition function of the system and a Hubbard Statonovich Transformation

The presented concept is originally known from the work of Kramers and Wannier in 1941, in which they showed that the two-dimensional Ising Model could be transformed into another two-dimensional Ising Model but with inverted temperature behaviour. This allowed them to gain new insights into the structure of the model. Since this kind of transformation can be applied to almost every abelian theory yielding a new set of variables ('disorder variables'), that have small fluctuations in the regime in which the original variables had high fluctuations and vice versa, it has been used for a wide field of theories (for further information on the general form of Duality Transformations see [24]).

4.2 Introduction of the Partition Function

Looking at our system we immediately see that we do not have a constant particle number, because the vortices and antivortices can appear and disappear together. This means that we have to use a grand canonical partition function to statistically describe it. Since for N vortices there are N statistical subsystems contributing, the overall partition function has to consider all vortex configurations that yield N vortices, while multiplying all present vortex subsystems. For a quantum field theory the time dependent form of this 'partition function' takes the form:

$$Z = \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp[-S(\mathbf{x}^{(i)})] \tag{4.1}$$

Here the factor in front of the integral sums up all possible vortex configurations or in the model magnetic flux combinations $j_M(\mathbf{x})$ for N vortices, while avoiding double counting through the $\frac{1}{N!}$ contribution .
Additionally we have to multiply the fugacity (absolute activity) ζ of the vortices to each $x^{(i)}$ integral. This physical variable describes the effective partial pressure of a gas in thermodynamics $\zeta = \exp(\frac{\mu}{kT})$ depending on the temperature T and chemical potential μ. Therefore in our model the fugacity will determine in which energy state the system is (see figure 2.4) with the phase transitions at its limits.

Now we only need to plug the action of the system into the general expression for a grand canonical system (equ. 4.1) to get to the starting point of the duality transformation:

$$\boxed{Z = \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp\left[-\frac{K}{2}\int \mathrm{d}^{(2+1)}x \widetilde{F}_\mu \widetilde{F}^\mu\right]} \tag{4.2}$$

4.3 Simplifying the Partition Function: Hubbard Stratonovich Transformation

At the moment we cannot perform the integration over the vortex positions $\mathbf{x}^{(i)}$ easily, since the dependency is in the electromagnetic field vector, which appears quadratic in the partition function. Therefore we want to perform a so called Hubbard Stratonovich Transformation, that shifts the square to an auxiliary field b_μ by completing the square and performing a Gaussian integral. The complete transformation procedure is shown in Appendix A.
The resulting general formula of the Hubbard Stratonovich Transformation says, that for $a > 0$:

$$\exp[-\frac{a}{2}x^2] = \sqrt{\frac{1}{2\pi a}}\int_{-\infty}^{\infty} \mathrm{d}y \exp[-\frac{y^2}{2a} - ixy] \tag{4.3}$$

Using this to transform the partition function of the system (equ. 4.2) we get:

$$Z = \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp\left[-\frac{K}{2}\int \mathrm{d}^{(2+1)}x \widetilde{F}_\mu \widetilde{F}^\mu\right] \tag{4.4}$$

$$\propto \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \int \mathrm{d}b_\mu \exp\left[-\int \mathrm{d}^{(2+1)}x \left(\frac{b_\mu b^\mu}{2K} + ib_\mu \widetilde{F}^\mu\right)\right] \tag{4.5}$$

Comparing this to the general formula above there are two differences we have to check additionally:
1. The quadratic elements $\widetilde{F}_\mu$ or b_μ are vectors and not scalars. Since μ is squared over on both sides of the transformation-equation this is just a sum over the 2+1 dimensions. Therefore we can separate the summands, so that the transformation is equivalent to the one of a scalar.
2. The second difference is, that there is an additional integral over x in the exponential function. If we have a closer look at the calculation of the transformation in Appendix A we see that this could only influence the integration in the last step. But since it is still a Gaussian Integral, the additional integral can only change the result by a constant factor, which doesn't make a difference for the following calculations.

The important point of the transformation is, that the auxiliary field b_μ only depends on $\mathbf{x}$ and not on the vortex positions $\mathbf{x}^{(i)}$ so that we are left with a linear dependency on the vortex positions, which enables us to integrate them out in the following steps. To do that we need to use the magnetic flux (that can be imagined as a vortex density) from which we have constructed the electromagnetic field vector. If we look at this construction, we see that - because of the underlying Maxwell equations - we can collapse the integral with the Green function by adding a second derivation to $\widetilde{F}_\mu$. To achieve this we describe the auxiliary field b_μ as a gradient field of a scalar disorder potential ϕ in the next chapter.

4.4 Introduction of the Scalar Disorder Field

As explained above we want to describe our theory by a scalar disorder field. Therefore we write the auxiliary field b_μ as a gradient field $b_\mu = \partial_\mu \phi$ of a disorder potential ϕ. Generally order fields are used to describe systems with a global symmetry that is broken spontaneously, so that in the resulting ordered state the order parameter has a non-vanishing expectation value. The disorder field is the counterpart of this with the expectation value vanishing in the broken symmetry phase and non-vanishing in the disordered phase. It has first been introduced by Leo Kadanoff in 1971 for the 2D Ising model. For more information on the concept of order- and disorder-fields see [25], [8].

Inserting the disorder field ϕ our partition function takes the form:

$$Z \propto \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \int \mathrm{d}(\partial_\mu \phi) \exp\left[- \int \mathrm{d}^{(2+1)}x \left(\frac{\partial_\mu \phi \partial^\mu \phi}{2K} + i \partial_\mu \phi \widetilde{F}^\mu \right) \right] \tag{4.6}$$

The second summand in the exponential function is now the only part of the equation that is dependent on the vortex positions. Performing a partial integration the derivation can be pulled in front of $\widetilde{F}^\mu$ so that we get the desired form. Now - as constructed - it can be simplified by plugging in its definition (equ. 3.23) and collapsing the greens function.

$$\begin{aligned} \int \mathrm{d}^{(2+1)}x i \partial_\mu \phi \widetilde{F}^\mu &= \int \mathrm{d}^{(2+1)}x i \partial_\mu \phi (-4\pi) \partial^\mu \int \mathrm{d}^{(2+1)}x' G_R^{(2+1)}(t-t', \mathbf{x}-\mathbf{x}') j_M(t', \mathbf{x}') \\ &= \int \mathrm{d}^{(2+1)}x i \phi \partial_\mu \partial^\mu (4\pi) \int \mathrm{d}^{(2+1)}x' G_R^{(2+1)}(t-t', \mathbf{x}-\mathbf{x}') j_M(t', \mathbf{x}') \\ &= \int \mathrm{d}^{(2+1)}x i \phi (4\pi) \int \mathrm{d}^{(2+1)}x' \delta^{(2+1)}(t-t', \mathbf{x}-\mathbf{x}') j_M(t', \mathbf{x}') \\ &= \int \mathrm{d}^{(2+1)}x i \phi (4\pi) j_M(t, \mathbf{x}) \end{aligned}$$

In the second to last step we utilized that the differential operator $\partial_\mu \partial^\mu (4\pi)$ is dependent on x and not x' so that we can pull it into the second integral and combine it with the greens function to get a delta functional.

Finally we perform another partial integration and use $\phi \longrightarrow \frac{1}{4\pi}\phi$ to get to the easiest possible form of the partition function we can later apply in the Duality Transformation. Additionally we use the new constant $K' = 16\pi^2 K$ from here on in the calculations.

$$\boxed{Z \propto \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \int \mathrm{d}(\partial_\mu \phi) \exp\left[- \int \mathrm{d}^{(2+1)}x \left(\frac{\phi(-\partial^2)\phi}{16\pi^2 K} + i \phi j_M(\mathbf{x}) \right) \right]} \tag{4.7}$$

4.5 Duality Transformation

Having reduced the partition function to contain just a linear dependence of j_M as sole part with an influence of the vortex positions $\mathbf{x}^{(i)}$ the only thing we are left to do now to finish the duality transformation is performing their integrals. The calculations in the second half of this section are parallel to those in Schaposnik et al. in *Pseudoparticles and confinement in the two-dimensional Abelian Higgs model* [26].

We start with the fully reduced partition function from the last chapter:

$$Z \propto \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \int \mathrm{d}(\partial_\mu \phi) \exp\left[-\int \mathrm{d}^{(2+1)}x \left(\frac{\phi(-\partial^2)\phi}{K'} + i\phi j_M(\mathbf{x})\right)\right] \tag{4.8}$$

Now we can move the first summand in the exponential function in front of the $\mathbf{x}^{(i)}$ integrals, since ϕ does not depend on the positions of the vortices.
Additionally we plug in the magnetic flux : $j_M(\mathbf{x}) = \sum_i q^{(i)} \delta^{(2+1)}(\mathbf{x} - \mathbf{x}^{(i)})$.

$$\begin{aligned}
Z \propto & \int \mathrm{d}(\partial_\mu \phi) \exp\left[-\int \mathrm{d}^{(2+1)}x \left(\frac{\phi(-\partial^2)\phi}{K'}\right)\right] \\
& \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp\left[-\int \mathrm{d}^{(2+1)}x \left(i\phi j_M(\mathbf{x})\right)\right] \\
& = \int \mathrm{d}(\partial_\mu \phi) \exp\left[-\int \mathrm{d}^{(2+1)}x \left(\frac{\phi(-\partial^2)\phi}{K'}\right)\right] \\
& \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp\left[-i\int \mathrm{d}^{(2+1)}x \left(\sum_i q^{(i)} \delta^{(2+1)}(\mathbf{x} - \mathbf{x}^{(i)})\phi\right)\right]
\end{aligned}$$

To shorten the calculation, in the following we will only look at the $\mathbf{x}^{(i)}$ dependent part of the partition function. We find that:

$$\begin{aligned}
& \sum_{N\{j_M(\mathbf{x})\}} \frac{\zeta^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mathbf{x}^{(i)} \exp\left[\int \mathrm{d}^{(2+1)}x \left(-i\sum_i q^{(i)} \delta^{(2+1)}(\mathbf{x} - \mathbf{x}^{(i)})\phi\right)\right] \\
& = \sum_N \frac{\zeta^N}{N!} \left[\int \mathrm{d}^{(2+1)}x (\exp(i\phi) + \exp(-i\phi))\right]^N
\end{aligned}$$

First we collapsed the integrals over $x^{(i)}$ with the delta functions and restricted the monopole charges $q^{(i)}$ to ± 1 neglecting higher order vortices, so that we get N factors of $\exp(\pm i\phi)$.

Then using the fact that the sum over all vortex configurations $(N\{j_M(\mathbf{x})\})$ - which is taken into account here through the binomial coefficient $\binom{N}{k}$ - can be written by the binomial theorem.

$$(x+y)^N = \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k} \tag{4.9}$$

We can reduce it to a sum over the number of vortices N, hereby changing the product over the exponential functions into a sum.

In the next step we see that the resulting sum is exactly the series expansion of a cos function and can be written as $\cos(\phi(x))$.

$$\begin{aligned} ... &= \sum_N \frac{\zeta^N}{N!} \left[\int \mathrm{d}^{(2+1)} x (\exp(i\phi) + \exp(-i\phi)) \right]^N \\ &= \sum_N \frac{\zeta^N}{N!} \left[\int \mathrm{d}^{(2+1)} x 2 \cos(\phi(\mathbf{x})) \right]^N \end{aligned}$$

Finally we can collapse everything back into a new exponential function using the sum that is left of the construction of the grand canonical partition function.

$$\begin{aligned} ... &= \sum_N \frac{\zeta^N}{N!} \left[\int \mathrm{d}^{(2+1)} x 2 \cos(\phi(\mathbf{x})) \right]^N \\ &= \exp \left[2\zeta \int \mathrm{d}^{(2+1)} x \cos(\phi(\mathbf{x})) \right] \end{aligned}$$

If we now combine the result of this calculation with the rest of the partition function we get :

$$\boxed{Z \propto \int \mathrm{d}(\partial_\mu \phi) \exp \left[\int \mathrm{d}^{(2+1)} x \left(- \left(\frac{\phi(-\partial^2)\phi}{K'} \right) + 2\zeta \cos(\phi(\mathbf{x})) \right) \right]} \tag{4.10}$$

This is a new but equivalent partition function which instead of summing over all vortex positions is now dependent on the disorder field ϕ and its integral, that we got from the Hubbard Stratonovich Transformation. If we take the action back out of the partition sum, we are left with a statistically equivalent or 'dual' action, the resulting Dual Sine-Gordon Action:

$$\boxed{S = \int \mathrm{d}^{(2+1)} x \left[\left(\frac{\phi(-\partial^2)\phi}{K'} \right) - 2\zeta \cos(\phi(\mathbf{x})) \right]} \tag{4.11}$$

4.6 Discussion of our Result: Sine-Gordon Theory

Starting with a monopole model for vortices in 2+1 d we showed through a duality transformation, that statistically we can describe vortex behaviour with a Sine-Gordon potential consisting of a relativistic kinetic term and a cosine potential. The amplitude of this potential is proportional to the fugacity of the vortices.

$$S = \int \mathrm{d}^{(2+1)}x \left[\left(\frac{\phi(-\partial^2)\phi}{K'} \right) - 2\zeta \cos(\phi(\mathbf{x})) \right] \tag{4.12}$$

A plot of the cosine potential $V(\phi) = 2\zeta \cos(\phi(\mathbf{x}))$ in dependency of the disorder field ϕ can be seen in figure 4.2.

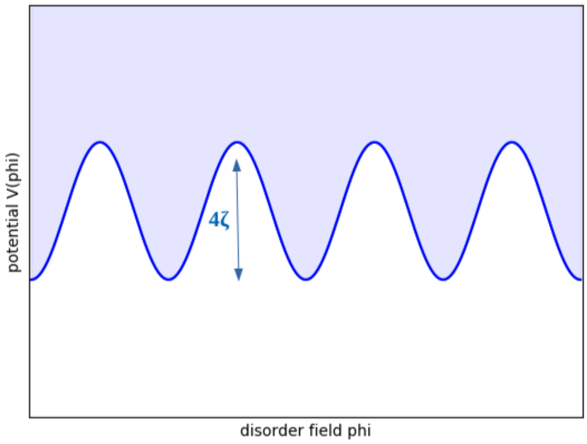


Figure 4.2: Plot of the cosine potential of the Sine-Gordon Theory in dependency of the scalar disorder field ϕ. The amplitude of the cosine is proportional to the fugacity ζ.

In comparison to the electromagnetic model in the beginning we now have a simple description through one scalar field, the disorder field, that depends solely on the overall vortex activity of the system (simply put: the more free vortices are present the bigger ϕ). Depending on the fugacity of the vortices that changes with the temperature the potential well depth changes, which justifies the different vortex behaviour near the BKT-Transition. The full interpretation of the consequences of this theory on the vortex behaviour can be found in chapter 6 of this thesis.
The Sine-Gordon Model as a nonlinear wave model has many applications, since its Euler Lagrange equation (the Sine-Gordon Equation) produces soliton solutions. Because of this it is used in many different fields of physics from astrophysics to condensed matter and high energy physics. For further information on the applications of the Sine-Gordon Model see J.Cuevas-Maraver et al. *The sine-Gordon Model and its Applications: From Pendula and Josephson Junctions to Gravity and High-Energy Physics*[27].

Part III

Universal Scaling Behaviour and Interpretation of the Dual Sine-Gordon Theory

5 Approximation as a ϕ^4 Theory

Next we want to find the universality class of the Sine-Gordon Theory we arrived at in our calculations, so that we can deduce the scaling exponents. For the full theory this is quite difficult, since the cosine function in the potential is an infinite potential series. Therefore we start by only considering it up to the quartic term, which is possible for small values of the disorder field ϕ.
The universal scaling of the resulting approximated ϕ^4 theory has been studied before, for example by Asier Piñeiro Orioli et al. in *Universal self-similar dynamics of relativistic and nonrelativistic field theories near nonthermal fixed points* [28], whose findings we are going to follow in this chapter to determine the approximate universal scaling exponents of our model.

From here on we are using $c = 1$ to shorten the presented calculations.

5.1 Approximation as a Relativistic Massive Scalar Field Theory

Looking at the dual Sine-Gordon Action we see that the potential takes the form of a cosine function. To obtain a universal scaling behaviour for this potential is rather difficult, since the cosine is equivalent to an infinite potential series. In the following chapter we choose the approach to only consider it up to the quartic order to be able to find the universality class of our system. This is possible for small values of the disorder field ϕ in the argument of the cosine function.

The potential series depiction of the cosine is:

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \tag{5.1}$$

Next we plug the approximation of the cosine into the Lagrangian corresponding to the Sine-Gordon action (4.11) we found in the last chapter.
We get:

$$\mathcal{L} = \left(\frac{\partial_\mu \phi \partial^\mu \phi}{K'}\right) - 2\zeta \cos(\phi(\mathbf{x})) \tag{5.2}$$

$$= \left(\frac{\partial_\mu \phi \partial^\mu \phi}{K'}\right) - 2\zeta + \zeta \phi^2 - \frac{\zeta}{12}\phi^4 + O(\phi^6) \tag{5.3}$$

To simplify this expression we drop the constant summand -2ζ which we can always do in a Lagrangian, since as a gauging it doesn't change the equations of motion and consequently also doesn't change the physical behaviour of the system. We are left with a new Lagrangian $\mathcal{L}_{O(4)}$ that is cut off at the fourth order:

$$\boxed{\mathcal{L}_{O(4)} = \left(\frac{\partial_\mu\phi\partial^\mu\phi}{K'}\right) + \zeta\phi^2 - \frac{\zeta}{12}\phi^4} \tag{5.4}$$

This is a standard relativistic Lagrangian describing a scalar potential consisting of a kinetic term $T = \left(\frac{\partial_\mu\phi\partial^\mu\phi}{K'}\right)$, a mass term $\zeta\phi^2$ and an interaction term $\frac{\zeta}{12}\phi^4$.
In figure 5.1 one can see a plot of the potential portion V of the Lagrangian. From the definition of the Lagrangian $\mathcal{L}_{O(4)} = T - V$ we find it to be:

$$V = -\zeta\phi^2 + \frac{\zeta}{12}\phi^4 \tag{5.5}$$

The combination of the quadratic and the quartic term results in a symmetric double well potential, where the two minima are closer together and less deep than in the original cosine potential.

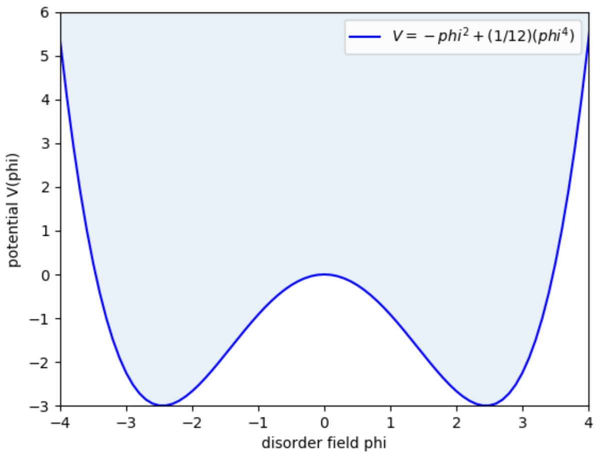


Figure 5.1: Plot of the approximation of the cosine potential up to quartic order for small values of the disorder field ϕ (setting the fugacity to ζ=1)

To find the equations of motion of this Lagrangian we are using the definition of the Euler Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{5.6}$$

Applied to the forth order Lagrangian $\mathcal{L}_{O(4)}$ we get the equation of motion for the approximated system:

$$\boxed{(\partial_\mu\partial^\mu - 2\zeta + \frac{\zeta}{3}\phi^2)\phi = 0} \tag{5.7}$$

5.2 Calculation of the Scaling Exponents

With this result we can now find the universal scaling class following the calculations of Asier Piñeiro Orioli et al. in *Universal self-similar dynamics of relativistic and non-relativistic field theories near nonthermal fixed points* [28].

In this paper two types of theories are considered, the non-relativistic complex Bose field which is described by the Gross-Pitaevskii equation and a relativistic massless scalar field theory with ϕ^4 interaction. Doing numerical simulations they found that the scaling of the relativistic theory fell into the same universality class as the one for the non relativistic Bose gas. In figure 5.2 the numeric results of Asier Piñeiro Orioli et al. for the universal scaling exponents can be seen, showing that they adopt the same values that are found numerically and analytically for the non-relativistic theory but differing from the values calculated analytically for the relativistic theory (resulting in $\beta = 1$). The reason for that is shown to be the generation of a mass gap in the mean field approximation of the relativistic theory.

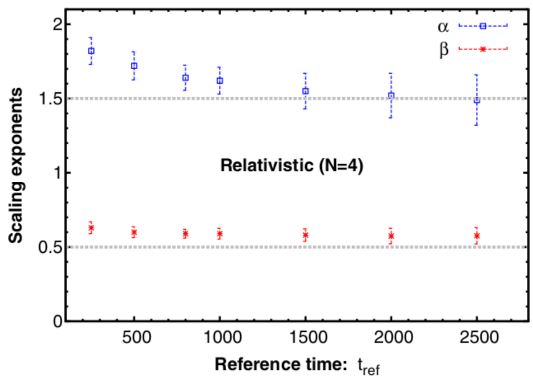


Figure 5.2: Plot of numerical results of Asier Piñeiro Orioli et al. for the universal scaling exponents α and β at different reference times for the relativistic four component theory. Figure taken from [28]

The approximated Lagrangian we constructed in the last chapter (equ. 5.7) belongs to a relativistic scalar field theory with a mass contribution and a ϕ^4 interaction term. This result tells us that our theory should fall into the same universality class as the one of a non-relativistic complex Bose field, described by the Gross-Pitaevskii equation. Therefore we can follow their calculations to conclude the expected values for the universal scaling exponents from a vertex resummed kinetic theory approach (chapter IV in [28]).

We start with the expression for a self similar evolution of the distribution function $f(t, \mathbf{p})$ for a spatially homogeneous and isotropic system (see equation 2.6). Here we are using f instead of n for the momentum distribution to be consistent with the naming in [28]) :

$$f(t, \mathbf{p}) = s^{\frac{\alpha}{\beta}} f(s^{-\frac{1}{\beta}} t, s\mathbf{p}) \tag{5.8}$$

In this term s is the scaling parameter and α and β are the scaling exponents, so that choosing $s^{-\frac{1}{\beta}} t = 1$ (or equivalent $s = t^{\beta}$) we get a time independent scaling function (equ. 2.6), as discussed in chapter 2.4 of this thesis. Using this we can rewrite the time evolution of the distribution function (the collision integral $C[f](t, \mathbf{p})$) by introducing a new function μ of scaling exponents:

$$C[f](t, \mathbf{p}) = \frac{\partial f(t, \mathbf{p})}{\partial t} = s^{-\mu} \frac{\partial f(s^{-\frac{1}{\beta}} t, s\mathbf{p})}{\partial t} \tag{5.9}$$

$$= t^{-\beta\mu} \frac{\partial f(1, t^{\beta}\mathbf{p})}{\partial t} = t^{-\beta\mu} \frac{\partial f(1, t^{\beta}\mathbf{p})}{\partial t} \tag{5.10}$$

$$= t^{-\beta\mu} C[f](1, t^{\beta}\mathbf{p}) \tag{5.11}$$

On the other hand we can evaluate the time evolution of the rescaled form with $\xi = t^{\beta}\mathbf{p}$:

$$\frac{\partial f(t, \mathbf{p})}{\partial t} = \frac{\partial (t^{\alpha} f(1, t^{\beta}\mathbf{p}))}{\partial t} \tag{5.12}$$

$$= \alpha(t^{\alpha-1}) f(1, t^{\beta}\mathbf{p}) + t^{\alpha} \frac{\partial (t^{\beta}\mathbf{p})}{\partial t} \nabla_{t^{\beta}\mathbf{p}} f(1, t^{\beta}\mathbf{p}) \tag{5.13}$$

$$= t^{\alpha-1}(\alpha + \beta\xi\nabla_{\xi}) f(1, \xi) \tag{5.14}$$

Comparing the exponents of these two equations we see that the collision integral for the rescaled distribution function is $C[f](1, \xi) = (\alpha + \beta\xi\nabla_{\xi}) f(1, \xi)$ so that the t dependency finally gives us the scaling relation:

$$\boxed{\alpha - 1 = -\beta\mu} \tag{5.15}$$

Assuming particle or energy conservation we can get further restraints for the scaling exponents. For particle conservation we set the particle number n to be constant and plug in the rescaled distribution function:

$$n = \int \frac{\mathrm{d}^d p}{(2\pi)^d} f(t, \mathbf{p}) = t^{\alpha - \beta d} \int \frac{\mathrm{d}^d \xi}{(2\pi)^d} f(1, \xi) \tag{5.16}$$

Next we do the same for the energy density using the scaling of frequency $\omega(\mathbf{p}) \propto \mathbf{p}^z$. For the non-relativistic model the dispersion relation is $\omega_p = \frac{\mathbf{p}^2}{2m}$ so that $z = 2$

$$\epsilon = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \omega(\mathbf{p}) f(t, \mathbf{p}) = t^{\alpha - \beta(d+z)} \int \frac{\mathrm{d}^d \xi}{(2\pi)^d} f(1, \xi) \tag{5.17}$$

This results in the scaling conditions:

$$\alpha = \beta d \quad \text{(particle conservation)} \tag{5.18}$$
$$\alpha = \beta(d+z) = \beta(d+2) \quad \text{(energy conservation)} \tag{5.19}$$

Now one could look at a perturbative kinetic theory to find the values for the scaling exponents, but this approach leads to negative values of α and β which does not match the numeric results. Instead we are also going to take vertex corrections into account, which can be done for an overoccupied regime (see chapter 2.4 of this thesis). Let us first look at the normal definition of the collision integral of the Quantum Boltzmann Equation (for two particles scattering into two particles $p, l \leftrightarrow q, r$) and its approximation :

$$C^{2\leftrightarrow 2}[f](t,\mathbf{p}) = \int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r})[(f_p+1)(f_l+1)f_q f_r - f_p f_l (f_q+1)(f_r+1)] \tag{5.20}$$
$$\approx \int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r})[(f_p+f_l)f_q f_r - f_p f_l(f_q+f_r)] \tag{5.21}$$

For the next to leading order vertex correction we consider a new form of $\Omega^{2\leftrightarrow 2}(p,l,q,r)$ by exchanging the constant scalar factor g with a time and momentum dependent factor g^2_{eff} containing the one-loop retarded self energy Π^R (for detailed calculation see chapter IV.C of [28]).

$$\int \mathrm{d}\Omega^{NLO}(p,l,q,r) = \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}^d r}{(2\pi)^d}(2\pi)^{d+1}\delta^{(d)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r}) \tag{5.22}$$
$$\delta(\omega_p+\omega_l-\omega_q-\omega_r)g^2_{eff}[f](t,\mathbf{p},\mathbf{q}) \tag{5.23}$$
$$g^2_{eff}(t,\mathbf{p},\mathbf{q}) = \frac{g^2}{|1+\Pi^R(t,\omega_p-\omega_q,\mathbf{p}-\mathbf{q})|^2} \tag{5.24}$$
$$\Pi^R(t,\omega,\mathbf{p}) = \lim_{\epsilon\to 0} g\int \frac{\mathrm{d}^d q}{(2\pi)^d} f(t,\mathbf{p}-\mathbf{q}) \tag{5.25}$$
$$\left[\frac{1}{\omega_p - w_{p-q} - \omega - i\epsilon} + \frac{1}{\omega_q - w_{p-q} + \omega + i\epsilon}\right] \tag{5.26}$$

Now we can find the scaling behaviour of g^2_{eff} by fist looking at the scaling of one-loop retarded self energy Π^R :

$$\Pi^R(t,\omega_p,\mathbf{p}) = s^{\frac{\alpha}{\beta}-d+2}\Pi^R(s^{-\frac{1}{\beta}}t,\omega_{sp},s\mathbf{p}) \tag{5.27}$$

Let us shortly explain the exponent of s:
The $\frac{\alpha}{\beta}$ comes from the scaling of f, the $-d$ from the scaling of the momentum integral and the $+2$ from the scaling of ω.

From the scaling equations (5.18) and (5.19) we see that $\frac{\alpha}{\beta} \geq d$ so that the one-loop retarded self energy Π^R becomes large in the infrared for fixed $s\mathbf{p}$ and therefore dominates the denominator in g_{eff}. Using that we can find the scaling of g_{eff}^2 in the infrared regime:

$$g_{eff}^2(t, \mathbf{p}, \mathbf{q}) = s^{-2(\frac{\alpha}{\beta}-d+2)} g_{eff}^2(s^{-\frac{1}{\beta}}t, s\mathbf{p}, s\mathbf{q}) \tag{5.28}$$

Now we know the scaling for all the parts of the collision integral, so that - to find the scaling behaviour of the collision integral in the infrared regime - we only have to put everything together. For the exponent of s we get the following contributions: First the cubic influence of f in the Boltzmann equation gives us $3\frac{\alpha}{\beta}$, in $\int \mathrm{d}\Omega^{NLO}(p, l, q, r)$ we get $-3d$ from the momentum integrals, $+d$ from the momentum delta functions, $+2$ from the frequency delta function and finally $-2(\frac{\alpha}{\beta} - d + 2)$ from g_{eff}^2. Combined this equals :

$$3\frac{\alpha}{\beta} - 3d + d + 2 - 2\frac{\alpha}{\beta} + 2d - 4 = \frac{\alpha}{\beta} - 2 \tag{5.29}$$

With that we find for the scaling of the collision integral in the overoccupied infrared regime:

$$C^{NLO}[f](t, \mathbf{p}) = s^{\frac{\alpha}{\beta}-2} C^{NLO}[f](1, t^{\beta}\mathbf{p}) \tag{5.30}$$

$$= t^{\alpha-2\beta} C^{NLO}[f](1, t^{\beta}\mathbf{p}) \tag{5.31}$$

Comparing this with the scaling of the collision integral we computed before we find immediately for the overoccupied infrared regime:

$$\boxed{\beta = \frac{1}{2}} \tag{5.32}$$

From this we can calculate α for the cases of particle and energy conservation:

$$\boxed{\alpha = \frac{d}{2} \text{ (particles)} \qquad \alpha = \tfrac{(d+2)}{2} \text{ (energy)}} \tag{5.33}$$

If we compare this again to numeric results e.g. in figure 5.2 (for d=3 and particle conservation) we see that they are in good agreement, but to arrive at these universal scaling exponents for the Sine-Gordon Theory we had to make various assumptions and approximations. Therefore in the next chapter we want to discuss to what extent this result can be used to make a statement about the universal scaling of vortex behaviour.

5.3 Interpretation of the Result and Problems of the Approximation

In the last chapter we determined the universal scaling of a Bose gas that is quenched to far from equilibrium initial conditions, so that vortices are generated, and then experiences a universal scaling toward the ordered phase of pair collapsed vortices. To do this we constructed a dual Sine-Gordon action describing the disorder field of the vortex activity. For the forth order approximation of the dual Sine-Gordon Model we found the universal scaling exponent $\beta = \frac{1}{2}$ belonging to the Gaussian fixed point. As we discussed in the introduction of this thesis we also expect a different scaling of the vortex behaviour $\beta = \frac{1}{5}$ for the anomalous fixed point which is much slower (see figure 1.2), which we did not find here. Thus in this chapter we want to go through all the approximations we made to get to this result and discuss whether they are justified.

The first approximation we made is, that the disorder field ϕ is small, so that we could reduce the cosine potential of the Sine-Gordon Theory to a standard relativistic scalar field theory with a mass term and a ϕ^4 interaction term. This requires that the vortex activity in the system is low, which seems like a reasonable assumption, if we want to model the scaling towards the pair collapse regime.
On the other hand we can immediately see that the approximated potential is now reduced to a double well potential and to be able to generate another well in our potential we need to take many more orders of ϕ in the cosine function into consideration (see figure 5.3).

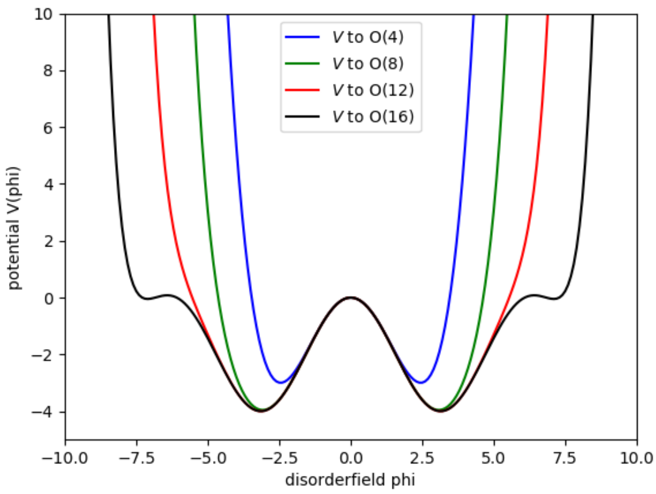


Figure 5.3: Plot of the disorder potential cut off at different orders. To get more wells in the potential one needs at least the 16th order in ϕ.

If we assume that jumping one well in the disorder potential means, that the vortex activity locally goes up by one vortex, this could indicate, that by only considering one well we ignore e.g. effects of more than two vortices meeting. In A. Hernández-Garduño et al. *Collisions of Four Point Vortices in the Plane* [29] it is shown, that if three vortices (two vortices and one antivortex or the other way round) meet, they form a semistable system orbiting one another which could e.g. be an effect that would slow the pair collapse down, possibly leading to a smaller value of β for a higher disorder regime.

In the calculation of the scaling we furthermore made the assumption, that the relativistic scalar system with a mass contribution has the same universality class as the nonrelativistic model descibed by the GPE equation, having the same universal scaling function and scaling exponents in the infrared regime [28]. To my knowledge this is justified by numerical simulations and theoretical considerations, but it would still be interesting to know more about the physical background of this observation.

Additionally we used the requirement, that the system is overoccupied in the infrared regime to make a vertex resummed approach in the scaling equations. As explained in chapter 2.4 of this thesis, this assumption is supported by the fact, that we quenched our system, so that the momentum distribution is strongly overpopulated below a maximum momentum scale and additionally the particle transport goes in the direction of low momenta.

Considering all of these approximations the fact that we ignored all orders above the forth in the disorder field seems to be the most likely reason for the scaling being faster that we expected it to be. Therefore in the next chapter we want to have a short look on the full theory discussing its physical meaning and its behaviour close to the phase transitions.

6 Effect of Higher Order Terms

In the following we are going to have a closer look at the full dual Sine-Gordon Theory we developed as a model for the disorder field of the vortices. First we are discussing the physical meaning of this result and comparing it to the forth order approximation from chapter 5. Then we will study the behaviour of the model at the limits of the temperature and compare it to the physical background from chapter 2.2 and 2.3 of this thesis describing quantum phase transition behaviour of the vortices.

6.1 Interpretation of the Full Theory

The first thing we need to get a physical understanding of the Sine-Gordon Theory is to discuss the meaning of the disorder field ϕ and the fugacity ζ .

The disorder field is a field with vanishing expectation value in the ordered phase, while it grows with disorder. In our context the natural interpretation of this is that the disorder field is some kind of overall vortex activity. Since the Sine-Gordon Model is dual to the monopole vortex model, meaning that the two models have equivalent partition sums, it is only a statistical statement about our system. Therefore we expect the disorder field $\phi(\mathbf{x})$ to describe the mean vortex activity at the position $\mathbf{x}$. Thus if the disorder field goes up 2π we would assume the creation of a vortex, or more exactly a higher mean vorticity of one vortex (and vice versa for an antivortex or the annihilation of a vortex). Here it is important to note that we rescaled the disorder field by 4π in the transformation process of the action, which could have an impact on the physical interpretation of the field.

The second physical variable that appears in the Sine-Gordon potential is the fugacity. It acts as the amplitude of the potential and can be seen as the 'thermodynamic activity' of the vortices. In figure 6.1 the influences of both parameters are shown.
In classical thermodynamics the fugacity is defined as

$$\zeta = exp(\frac{\mu}{kT}) \tag{6.1}$$

depending on the chemical potential μ, the Boltzmann constant k and the temperature T. Since the chemical potential is negative for a Bose gas the fugacity is growing for higher positive temperature.
For vortices it can also be written as a function of the vortex core energy E_C and a constant β :

$$\zeta = exp(-\beta E_C) \tag{6.2}$$

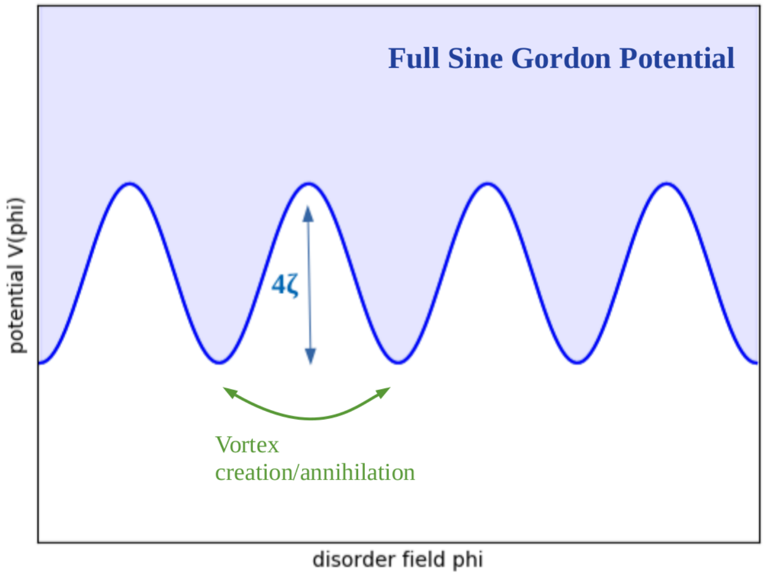


Figure 6.1: Plot of the full Sine-Gordon potential showing the regular potential wells for the disorder field jumping because of vortex creation or annihilation. The amplitude of the potential depends on the fugacity ζ.

If we look at the 'free vortex gas' as a Coulomb gas (which is a common approach for the Sine-Gordon Model: see e.g. [30], [31]) we can understand the fugacity as the thermodynamic activity separating the vortices as the counter part of the attractive force between vortices and antivortices.
In the next chapter we are using this physical understanding of the fugacity to have a look at the behaviour of the Sine-Gordon potential for different temperatures (meaning different fugacities) and comparing it to the effects we expect to happen at temperature limits.

6.2 Consequences at the Phase Transitions

To be able to gain insight into the temperature limit behaviour of the Sine-Gordon Theory we first need to calculate the corresponding values of the fugacity. To do that we use the thermodynamic definition of the fugacity from the last chapter. For positive temperatures we get:

$$\zeta \xrightarrow{T \to 0^+} exp(-\infty) \longrightarrow 0 \tag{6.3}$$

$$\zeta \xrightarrow{T \to +\infty} exp(0^-) = 1 \tag{6.4}$$

Next we calculate the fugacity for negative temperatures:

$$\zeta \xrightarrow{T \to 0^-} exp(\infty) \longrightarrow \infty \tag{6.5}$$

$$\zeta \xrightarrow{T \to -\infty} exp(0^+) = 1 \tag{6.6}$$

Now we can plot the potential for all these cases and compare it to the associated vortex states. The result can be seen in figure 6.2:

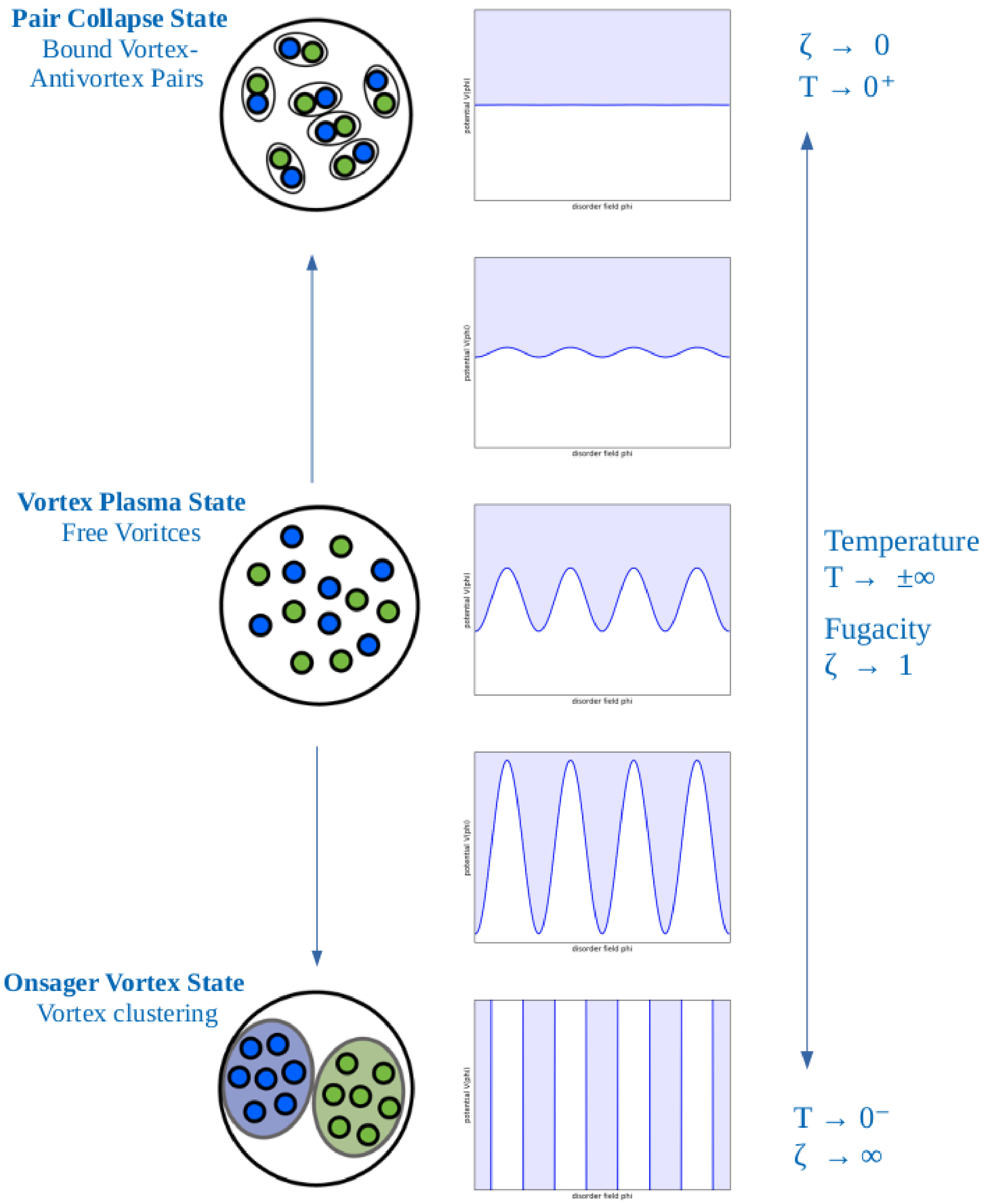


Figure 6.2: Plots of the Sine-Gordon potential for different temperatures T resulting in different fugacities ζ with the associated vortex state shown on the left side. Vortex state sketches adapted from [17].

Let us start with the Vortex Plasma State at $T = \pm\infty$, where the vortices are completely free. Here the fugacity is one so that we get a standard cosine potential. Comparing this we find that here we have free vortices giving a mean vorticity that sits in one of the potential wells of the cosine. It is possible to create and annihilate vortices by tunnelling into a neighbouring well. In this situation the thermodynamic activity counteracts the attraction of vortices and antivortices as described above resulting in a vortex plasma state.

Now if we look at the positive temperature case and cool down the system the vorticity also goes down and eventually in the limit $T \to 0^+$ to zero. Because of this the potential wells get less and less deep making it more probable to hop into a neighbouring well. Because of the attraction now bound vortex anti vortex states can occure with the counteraction thermodynamic activity going to zero. Therefore in the $T \to 0^+$ limit the potential is flat and all vortices are in a pair collapse state.

On the other side at negative temperatures we also start at the Vortex Plasma state for $T = -\infty$. If we now look at the limit $T \to 0^-$ with the fugacity going to infinity the potential wells get more and more deep making it less likely to change well. For the vortices this means that the probability of the mean vortex activity changing by vortex creation or annihilation goes to zero, which can be explained by the vortices clustering in groups of vortices or antivortices without the possibility to create vortex-antivortex pairs.

Overall it is important to state that this is only an interpretation of the resulting behaviour of the vortex potential and not a sufficient proof of the procedures described above. Nevertheless this discussion of the physical situation can be used as a starting point for further investigations on the phenomena.

7 Conclusion and Outlook

First let us summarize the results of the thesis. We started with building a magnetic monopole model for vortices on a plane. This model was sourced by a magnetic flux that cuts through the plane lying in an imaginary third space direction. Using the corresponding partition sum we formed a statistically equivalent dual theory from the resulting action - the Sine-Gordon Theory. Thus we arrived at a new dual action that is only dependent on a scalar disorder field instead of regarding all vortex positions. Subsequently we used a forth order approximation of the Sine-Gordon potential to deduce the universal scaling exponents of the vortex system and found that it falls into the same universality class as the classical GPE description. Finally we discussed the reliability of the approximations we made and examined the behaviour of the resulting potential at the known quantum phase transitions.

Altogether we can conclude that the biggest advantage of our result is the fact the the duality transformation makes it possible to lose the unnecessary information of the vortex positions in favour of a simple scalar potential. This makes the theory much more simple and easier to work with without neglecting any important properties. On the other hand an in depth understanding of the nature of the disorder potential is needed to be able to fully interpret the physical meaning of the results. Therefore the next step to further develop the theory could be to gain more insight on the background of order and disorder potentials.

Furthermore we could not yet produce the slow anomalous scaling behaviour we expected from the theory. As discussed in chapter 5.3 there are quite a few approximations going into the scaling result, probably the biggest one being the forth order approximation of the potential. It is likely that this simplification leads to the neglect of effects of more than two vortices meeting which could become important for higher disorder. If there would be stable three vortex groups occurring, it could be a reason for the pair collapse of the vortices to be suppressed. For future research it seems very promising to consider those effects and calculate their influence on the universal scaling exponents, in the best case by calculating the universal scaling for the full Sine-Gordon Theory.

Another interesting topic would be to look deeper into the meaning and impact of the fugacity. As we discussed in chapter 6.2 it seems to be possible to qualitatively explain the formation of the Pair Collapse vortex state and also the Onsager vortex state at the temperature limits by observing the behaviour of the Sine-Gordon potential dependent on the fugacity.

In this thesis the occurrences for the formation of the Onsager vortex state were only shortly discussed, leaving out a analytical examination of this quantum state transition. A more precise examination of the results from the Sine-Gordon Theory for the Onsager vortices could proof to be interesting.

Altogether the dual Sine-Gordon Theory seems to be a promising tool for further research on topological defect behaviour and its universal scaling and opens a new approach to gain physical insight into the topic. Since it is such an easy scalar theory focussing only on the statistical vortex behaviour it gives us the option to analytically investigate much further and with that to get a better understanding of the physic behind these phenomena.

Part IV

Appendix

A Hubbard Stratonovich Transformation - Derivation of the Formalism

In section 2.2 we use a Hubbard Stratonovich transformation to get rid of the quadratic $\tilde{F}_\mu$ term in the Lagrangian and introduce an auxiliary field b_μ.
To achieve this we use for $a > 0$:

$$\exp[-\frac{a}{2}x^2] = \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}y \exp[-\frac{y^2}{2a} - ixy] \tag{A.1}$$

This can be shown by first completing the square in the exponent in the right integrand and than performing the gaussian integral we are left with.

$$\begin{aligned}
&\sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}y \exp[-\frac{y^2}{2a} - ixy] \\
&= \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[\left(\frac{iy}{\sqrt{2a}} - \sqrt{\frac{a}{2}}x\right)^2 - \frac{a}{2}x^2\right] \\
&= \sqrt{\frac{1}{2\pi a}} \exp[-\frac{a}{2}x^2] \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[\left(\frac{iy}{\sqrt{2a}} - \sqrt{\frac{a}{2}}x\right)^2\right] \\
&= \sqrt{\frac{1}{2\pi a}} \exp[-\frac{a}{2}x^2] \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[-\frac{1}{2a}\left(y + aix\right)^2\right] \\
&= \sqrt{\frac{1}{2\pi a}} \exp[-\frac{a}{2}x^2] \int_{-\infty}^{\infty} \mathrm{d}u \exp\left[-\frac{1}{2a}u^2\right] \\
&= \sqrt{\frac{1}{2\pi a}} \exp[-\frac{a}{2}x^2] \sqrt{2a\pi} = \exp[-\frac{a}{2}x^2]
\end{aligned}$$

B Lists

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C Bibliography

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) ....